Algebraic Topology
Notes

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These notes are a combination of notes from Allen Hatcher's book "Algebraic Topology" and Prof. Ciprian Manolescu's lectures from MATH 215a at Stanford University and my own.

Introduction

Our goal is to develop algebraic invarianees associated with topological spaces.

We will look at

(1) Fundamental Group:

$$\pi_1(X) = \{\text{loops in } X\} / \text{homotopy}$$

(2) Homology Group:

$H_n(X)$, $n \in \mathbb{N}$ and abelian

Intuitively, they count "holes" in X

(3) Cohomolog Group:

$H^n(X) =$ Dual to $H_n(X)$

$\bigoplus_n H^n(X)$ is a ring!

Basic Constructions

Def : Homeomorphism

Let X and Y be topological spaces.
$f : X \to Y$ is a homeomorphism if f is bijective and both f and f^{-1} are continuous.
We say $X \cong Y$.

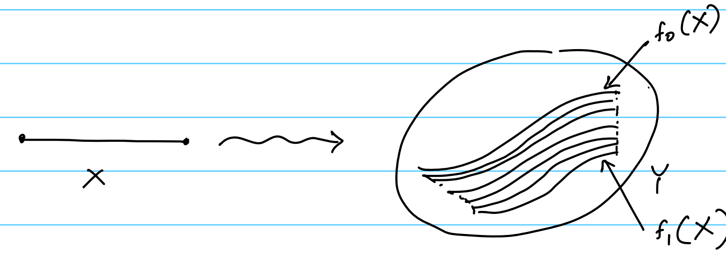
Def : Homotopy

A family of maps, $f_t : X \to Y$ where $t \in I = [0,1]$ s.t the associated map $F : X \times [0,1] \longrightarrow Y$ given by $F(x,t) = f_t(x)$ is continuous.

Two maps $f_0, f_1 : X \to Y$ are homotopic if there exists a homotopy $F : X \times [0,1] \to Y$ s.t

$$F(x,0) = f_0(x) \quad \forall x \in X$$
$$F(x,1) = f_1(x)$$

We say $f_0 \simeq f_1$.



Def : Homotopy Equivalence

A map $f : X \to Y$ is a homotopy equivalence if $\exists\, g : Y \to X$ s.t $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$.
We say the spaces X and Y are homotopy equivalent and $X \simeq Y$.

$\to$ can prove easily that this is an equivalence relation.

Examples of homotopy equivalence

(1) $\mathbb{R}^n \simeq$ a point (even though $\mathbb{R}^n \not\cong$ a point)

infinite finite

Why?

$f : \mathbb{R}^n \longrightarrow \{0\}$

and take $g : \{0\} \longrightarrow \mathbb{R}^n$ by $g(0) = 0$

Then $f \circ g = id_{\{0\}}$ and $(g \circ f)(x) = 0 \quad \forall x \in \mathbb{R}^n$

Now $g \circ f \sim id_{\mathbb{R}^n}$ by $f_t(x) = tx$ where $f_0 = 0$ and $f_1 = id_{\mathbb{R}^n}$

(2) $D^n = \{x \in \mathbb{R}^n \mid \|x\| \leq 1\} \simeq$ a point

$B^n = \{x \in \mathbb{R}^n \mid \|x\| < 1\} \simeq$ a point

Def : Contractible

We say the space X is contractible if $X \simeq$ a point

$\Updownarrow$

Equivalent definition : the identity map of X is null homotopic

homotopic to a constant map.

i.e $id_X \simeq$ constant map

Def : Retractions

Let X be a space and let $A \subset X$.

Then, a retraction is a map $r : X \longrightarrow X$ s.t

$r(X) = A$ and $r|_A = id_A$.

Def : Deformation Retraction

A deformation retraction of X onto a subspace A is a family of maps $f_t : X \longrightarrow X$, with $t \in I$ s.t

$f_0 = id_X$ and $f_1(X) = A$ and $f_t|_A = id_A$ for $\forall t \in I$.

The family f_t must also be continuous

$\rightarrow$ an example of a homotopy from id_X to a retraction of X onto $A \subset X$.

$\rightarrow$ in this case, $\boxed{A \simeq X}$ as $f_0 : A \hookrightarrow X$ by id_X

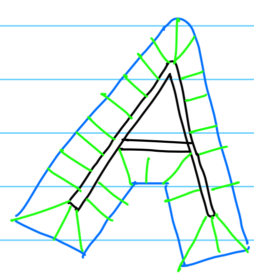
$f_1 : X \longrightarrow A$ as above

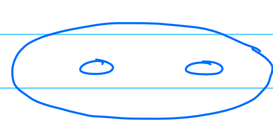
then $f_0 \circ f_1 \simeq id_X$ (since $f_0 \circ f_1 = f_1 \simeq f_0 = id_X$)

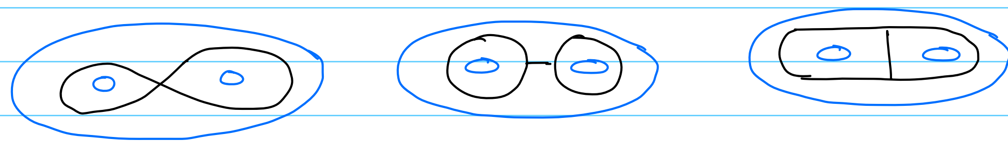
and $f_1 \circ f_0 = id_A$

Examples of deformation retraction:

(1)



(2) Look at deformations of 



(3) $X = \mathbb{R}^2 - \{0\}$. $A = S^1$

Use $F(x,t) = (1-t)x + t\dfrac{x}{\|x\|}$.

$X \rightsquigarrow x$

Proposition:

If X def. retracts to a point $x \in X$, then for any $U \subset X, x \in U$. $\exists\, V \subset U$ with $x \in V$ s.t the inclusion map $V \hookrightarrow U$ is nullhomotopic.

nullhomotopic $\longrightarrow$ homotopic to constant map

Def: Deformation Retraction in the weak sense:

Let $A \subset X$.

Then, there is the homotopy $f_t : X \longmapsto X$ s.t $f_0 = id_X$ and $f_1(X) \subset A$ with $f_t(A) \subset A$, $\forall t \in I$.

Lemma:

If X deformation retracts to A in the weak sense, then the inclusion $A \hookrightarrow X$ is a homotopy equivalence.

Proof:

Let the weak def. ret be f_t.

Let $\iota : A \longmapsto X$ by inclusion i.e $\iota(a) = a$, $\forall a \in A$.

Then, $(\iota \circ f_1)(x) = \iota(f_1(x)) = f_1(x)$, $\forall x \in X$

But $f_1 \simeq f_0 = id_X$

So, $i \circ f \simeq id_X$

Also, $(f_1 \circ \iota)(a) = f_1(a)$ ~~and~~ $\forall a \in A$

But $f_1|_A \simeq f_0|_A = id_X|_A = id_A$

$\Rightarrow f_1 \circ \iota \simeq id_A$

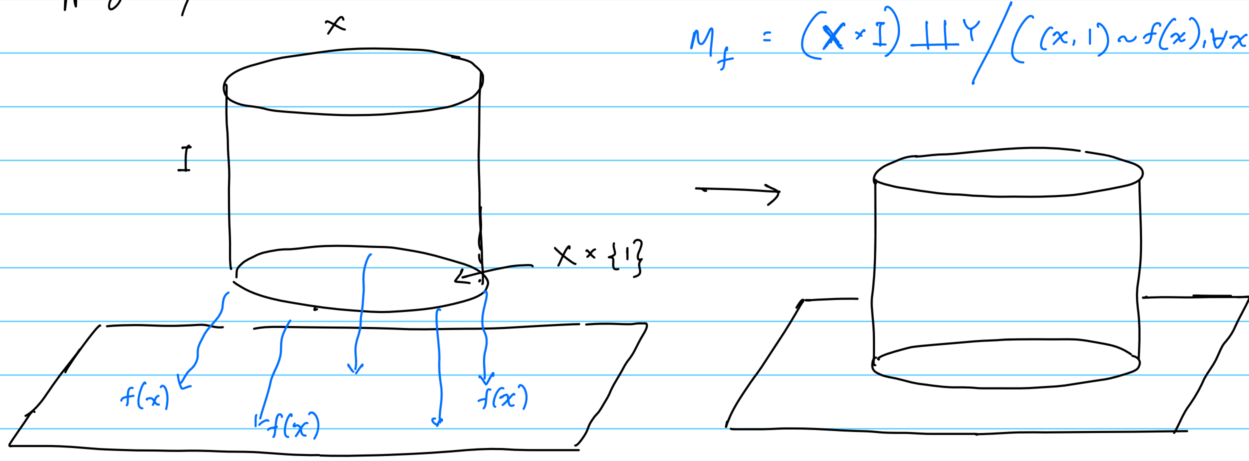
Def: Mapping Cylinder ("the structure through which the def. retraction occurs")

For a map $f: X \to Y$, the mapping cylinder M_f is the quotient space of the disjoint union $(X \times I) \amalg Y$ obtained by the equivalence $(x, 1) \in X \times I \sim f(x) \in Y$

Makes the endpoint of the deformation equivalent to the image of the map.

Mapping cylinders are continuous.

$$M_f = (X \times I) \amalg Y / ((x,1) \sim f(x), \forall x \in X)$$



Def: Homotopy relative to A (homotopy rel. A)

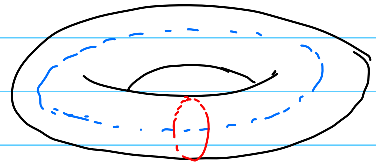
A homotopy $f_t : X \to Y$ whose restriction to a subspace $A \subset X$ is independent of t.

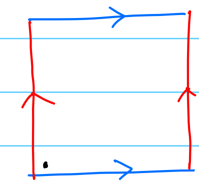
In other words, f_t is a homotopy and $f_t|_A$ is independent of t.

$\to$ def. retraction of X onto A is a homotopy rel. A from id_X to a retraction of X onto $A \subset X$.

Cell Complexes

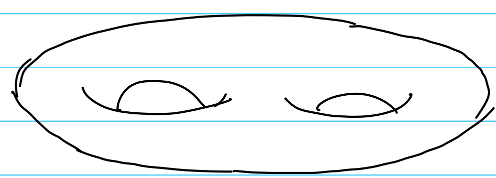
Examples:

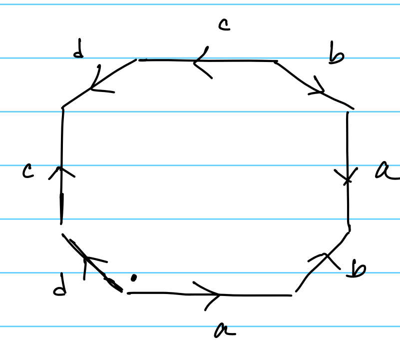




The torus $S^1 \times S^1$ can be constructed from the square

Generally, an orientable surface M_g of genus g can be constructed from a polygon of $4g$ sides, by identifying pairs of edges.





2 cell: interior of a polygon which is an open disk
1 cell: an open interval like $(0,1)$
3 cell: an open ball.
n-cell: open-disk

$S^{n-1} = \{x \in \mathbb{R}^n \mid \|x\| = 1\}$

Def : Cell Complex (or CW complex)

A space constructed as follows:

(1) Start with discrete set X^0 → the points are 0-cells

(2) Inductively, form the n-skeleton X^n from X^{n-1} by attaching n-cells e^n_α via maps

$$\varphi_\alpha : S^{n-1} \to X^{n-1}.$$

lives in $\mathbb{R}^n$ & is the boundary of D^n !

→ So, X^n is the quotient space of $X^{n-1} \sqcup_\alpha D^n_\alpha$ under the equivalence $x \sim \varphi_\alpha(x)$ $\forall x \in \partial D^n_\alpha$

X^{n-1}: (n-1)-skeleton; D^n_α: n-disks

i.e. attach boundaries of the n-disk to the (n-1)-skeleton

$\therefore X^n = X^{n-1} \bigsqcup_\alpha e^n_\alpha$ where e^n_α is an open n-disk

(3) Either stop this induction at a finite stage and set $X = X^n$ for $n < \infty$

or continue indefinitely, setting

$$X = \bigcup_n X^n$$

↳ in this case, X has the weak topology:

$A \subset X$ is open iff $A \cap X^n$ is open in X^n for each n

Vocabulary:

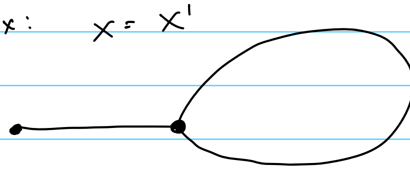
① X^n → n-skeleton

② Dimension of X → largest n s.t. an n-cell exists

Examples of Cell Complexes :

(1) 1-dimensional cell complex : $X = X^1$

multigraphs

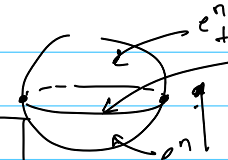


(2) The sphere S^n has a cell complex with two cells, e^0 and e^n. where e^n is attached by $\varphi : S^{n-1} \to e^0$

$\therefore$ S^n is being regarded as the quotient space $D^n / \partial D^n$

$S^n = e^0 \cup e^n$.

Alternatively,



$S^n = S^{n-1} \cup e^n_+ \cup e^n_-$
$= e^0_+ \cup e^0_- \cup \cdots \cup e^n_+ \cup e^n_-$

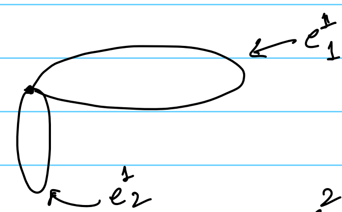
$\therefore S^\infty = \bigcup_n S^n$

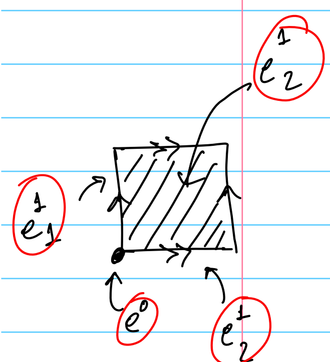
(3) Cell Complex of a torus :

Step 1 : X^0 is just 1 point $\longrightarrow \bullet \leftarrow e^0$

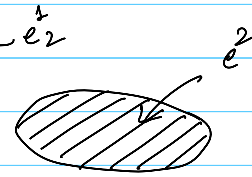
Step 2 : Attach two 1-cells to this point

$X^1 =$





Step 3 : Attach a disk



to X^1 by attaching its boundary to X^1.

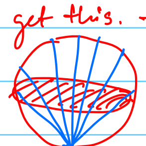
(4) Real Projective Space, $\mathbb{RP}^n$

$(\mathbb{R}^{n+1} - \{0\}) / (v \sim \lambda v,\ \forall v \in \mathbb{R}^{n+1}.\ \lambda \neq 0)$

→ Restricting to vectors of length 1, $S^n / (v \sim -v)$

$\Rightarrow$ D^n with antipodal points of ∂D^n identified

To get this, think of



i.e. for the upper hemisphere's points, find where the line to south pole intersects with D^n

∂D^n with antipodal points equivalent is $\mathbb{RP}^{n-1}$

$\therefore$ $\mathbb{RP}^n$ can be formed from $\mathbb{RP}^{n-1}$ by attaching an n-cell. and the attaching map $\varphi : S^{n-1} \longrightarrow \mathbb{RP}^{n-1}$

$\therefore$ $\mathbb{RP}^n$ has the cell complex structure $e^0 \cup e^1 \cup \cdots \cup e^n$ with one cell e^i in each dimension $i \leq n$.

(5) <u>Complex Projective Space. $\mathbb{CP}^n$</u>

Space of all complex lines through the origin in $\mathbb{C}^{n+1}$

ie $\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - \{0\}\right) / \left(v \sim \lambda v, \ \forall v \in \mathbb{C}^{n+1}, \lambda \neq 0\right)$

Equivalent to $S^{2n+1} / (v \sim \lambda v, \ |\lambda| = 1)$ $\quad (S^{2n+1} \subset \mathbb{R}^{2n+2} \cong \mathbb{C}^{n+1})$

Equivalent to $D^{2n} / (v \sim \lambda v, \ v \in \partial D^{2n})$

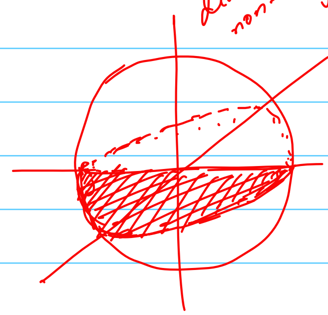
↳ <u>Why?</u>

$S^{2n+1} \subset \mathbb{C}^{n+1}$ → consider vectors in $\mathbb{C}^{n+1}$ whose last coordinate is ~~one~~ real and non-negative

in $\mathbb{R}^{2n+2}$, this kills one dimension & makes another dimension non-negative

These vectors are of the form $(w, \sqrt{1-w^2}) \in \mathbb{C}^n \times \mathbb{C}$ with $|w| \leq 1$

They form the graph of the function $w \longmapsto \sqrt{1-w^2}$ with $|w| \leq 1$, $w \in \mathbb{C}^n$



Note : $w \in \mathbb{C}^n$ and $|w| \leq 1 \Rightarrow w \in D^{2n}$

This is a disk D^{2n}_+ bounded by the spheres S^{2n-1}.

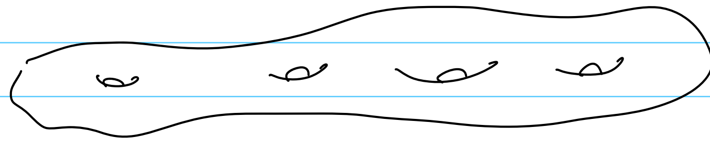
By adding another dimension and viewing them as $(w, 0) \in \mathbb{C}^n \times \mathbb{C}$, we ~~view th~~ view them as vectors in $(D^{2n}_+, 0)$ bounded by $S^{2n-1} \subset S^{2n+1}$

Now each vector in S^{2n+1} is equivalent to a vector in D^{2n}_+ by identifying $v \sim \lambda v$. In particular, if the last coordinate is zero, we have $v \sim \lambda v, \ \forall v \in S^{2n-1}$.

$\therefore$ $\mathbb{CP}^n$ is obtained from $\mathbb{CP}^{n-1}$ by attaching a cell e^{2n} using the attaching map $\varphi : S^{2n-1} \longrightarrow \mathbb{CP}^{n-1}$

$\therefore$ $\mathbb{CP}^n = e^0 \cup e^2 \cup \cdots \cup e^{2n}$ with cells only in even dimensions

(6)

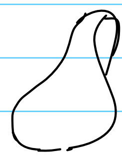


<u>Orientable surface</u> $\longrightarrow \Sigma_g \longrightarrow g$ holes / genus

Can be constructed from a $4g$ polygon

↳ Start with one e^0

(7)

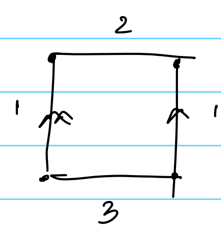


Non-orientable surface $\longrightarrow N_g$

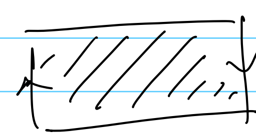
Eg: $N_2 \longrightarrow$ Klein bottle

$N_1 \longrightarrow \mathbb{RP}^2$

(8) <u>Annulus</u> :

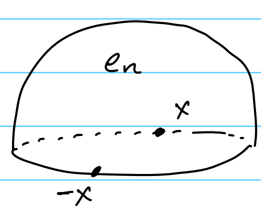


(9) <u>Mobius band</u>



(9) <u>$\mathbb{RP}^n$ revisited</u>

$$\mathbb{RP}^n = S^n / (x \sim -x, \forall x)$$



$\Rightarrow \mathbb{RP}^n = \mathbb{RP}^{n-1} \cup e^n$

$\therefore \mathbb{RP}^n = e^0 \cup e^1 \cup \cdots\cdots \cup e^n$

Then, $\mathbb{RP}^\infty = e^0 \cup e^1 \cup e^2 \cup \cdots\cdots = \bigcup_n \mathbb{RP}^n$

(10) <u>$\mathbb{CP}^n$ revisited</u>

$$\mathbb{CP}^n = \left(\mathbb{C}^{n+1} - \{0\}\right) \Big/ \left(x \sim \lambda x,\ \lambda \in \mathbb{C}^*\right)$$

norm 1

$$\therefore\ z \sim \frac{z}{\|z\|} \Rightarrow \mathbb{CP}^n \cong S^{2n+1} \Big/ \left(z \sim \lambda z,\ \lambda \in S^1\right)$$

Divide everything by x_1 i.e. last coordinate in $\mathbb{R}_{\geq 0}$

$$z = (\underbrace{z_0, \dots, z_n}_{w}, \underbrace{z_{n+1}}_{\sqrt{1-\|w\|^2}})$$

with $\|w\| \leq 1$

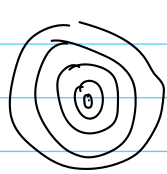
$$D^{2n}_+ = \text{graph}\left(w \longmapsto \sqrt{1-\|w\|^2}\right)$$

$$\therefore\ \mathbb{CP}^n = D^{2n}_+ \Big/ \left(w \sim \lambda w \text{ if } w \in S^{2n-1}\right)$$

$$= e^{2n} \cup \left(S^{2n-1}/(w \sim \lambda w)\right)$$

$$= \mathbb{CP}^{n-1} \cup e^{2n}$$

$$= e^0 \cup e^2 \cup \dots \cup e^{2n}$$

(11) 
Circles of Radii $\frac{1}{n}$ $\longrightarrow$ <u>NOT a cell complex</u>

Properties of CW Complexes

(1) They are normal ($\therefore$ also Hausdorff)

(2) Any finite cell complex is compact

(3) A compact subspace of a cell cx is contained in a finite subcomplex

(4) Closure finiteness $\longrightarrow$ The closure of each cell E meets only finitely many cells.

(5) Locally contractible:

$$\forall x \in X, \; U \ni x \text{ open}, \; \exists \, V \subset U \text{ with } x \in V \text{ s.t. } V \text{ is contractible}$$

(6)

Recall:
Top manifolds $\longrightarrow$ 2ND Countable, Hausdorff, locally Euclidean
Smooth manifolds $\longrightarrow$

Theorem: Every smooth manifold is homeomorphic to a cell complex.

Theorem: Every topological manifold is homotopy equivalent to a cell complex.

Theorem: Every top. manifold of dimension $\neq 4$ is homeomorphic to a cell complex (unknown in dim 4)

Def: Characteristic Map

Each cell e^n_α in a cell complex X has a characteristic map

$$\Phi_\alpha : D^n_\alpha \longmapsto X$$

which extends the attaching map φ_α and is a homeomorphism from the interior of D^n_α onto e^n_α

→ Φ_α is the composition

$$D^n_\alpha \hookrightarrow X^{n-1} \amalg_\alpha D^n_\alpha \overset{q}{\longmapsto} X^n \hookrightarrow X$$

the quotient map that defines X^n

Example of characteristic map:

(1) Recall: S^n can be constructed by two cells: e^0 and e^n ← just one point

where e^n is attached to e^0 by

$$\varphi_\alpha : S^{n-1} \longrightarrow e^0$$

Then, the characteristic map of e^n is

$$\Phi_\alpha : D^n_\alpha \longrightarrow S^n \text{ which collapses } \partial D^n_\alpha \text{ to } e^0$$

Def: Subcomplex

A subcomplex of a cell complex X is a closed subspace $A \subset X$ that is a union of cells of X.

→ As A is closed, for each cell in A,

the image of its characteristic map
the image of its attaching map $\Big\}$ contained in A

$\therefore$ A is a cell complex as well

Def: CW pair

A cell complex X and a subcomplex A forms a pair (X, A)

Example of subcomplex

→ Each skeleton, X^n, is a subcomplex.

→ in $\mathbb{R}P^n$ and $\mathbb{C}P^n$, the only subcomplexes are $\mathbb{R}P^k$ and $\mathbb{C}P^k$, $\forall k \leq n$

Properties of subcomplexes

(1) Closure of a collection of cells is a subcomplex.

(2) Any union and intersection of subcomplexes is a subcomplex.

Operations on Spaces

↳ Most of them preserves cell complexes ie if you start with CW complex, you get a CW complex.

① Products

$X, Y \longrightarrow$ cell complexes

$X \times Y \longrightarrow$ cell complex with the cells $\underbrace{e^m_\alpha}_{\text{cells of } X} \times \underbrace{e^n_\beta}_{\text{cells of } Y}$

② Quotients

$\longrightarrow \quad A \subset X \rightsquigarrow X/A$

Given (X, A) a CW pair, (cell complex, cell subcomplex)

the quotient space X/A also has a cell complex structure:

→ the cells of X/A are the cells of $X - A$ and a new 0-cell which is the image of A in X/A.

→ for a cell e^n_α of $X - A$ attached by $\varphi_\alpha : S^{n-1} \longrightarrow X^{n-1}$, the attaching map for the corresponding cell in X/A is the composition

$$S^{n-1} \longrightarrow X^{n-1} \longrightarrow X^{n-1}/A^{n-1}$$

Eg: (1) $D^n / S^{n-1} = S^n$

③ Wedge Sum (for based spaces)

Given spaces X and Y with chosen points $x_0 \in X$ and $y_0 \in Y$, the wedge sum $X \vee Y$ is the quotient of $X \amalg Y$ by identifying x_0 and y_0 to a single point

$$X \vee Y = X \amalg Y / (x_0 \sim y_0)$$

$\longrightarrow$ Examples: $S^1 \vee S^1 = \infty$

$\longrightarrow$ $\bigvee_\alpha X_\alpha$ for an arbitrary collection of spaces X_α: start with $\amalg_\alpha X_\alpha$ and then identify $x_\alpha \in X_\alpha$ to one point.

$\longrightarrow$ If X_α are cell complexes and the points x_α are 0-cells, then $\bigvee_\alpha X_\alpha$ is a cell complex because we obtain it from the cell complex $\amalg_\alpha X_\alpha$ and attach by collapsing a subcomplex to a point.

→ For a cell complex X, the quotient X^n / X^{n-1} is a wedge sum of n-spheres $\bigvee_\alpha S^n_\alpha$ with one sphere for each n-cell of X

Smash Product $X \wedge Y = (X \times Y) / ((x_0 \times Y) \cup (X \times y_0))$

Inside the product space $X \times Y$, there are copies of X and Y : $X \times \{y_0\}$ and $\{x_0\} \times Y$ for points $y_0 \in Y$ and $x_0 \in X$.

These copies of X and Y intersect only at (x_0, y_0) so their union can be identifies with the wedge sum $X \vee Y$

ie $(X \times \{y_0\}) \cup (\{x_0\} \times Y) = X \vee Y$
$= (X \amalg Y) / (x_0 \sim y_0)$

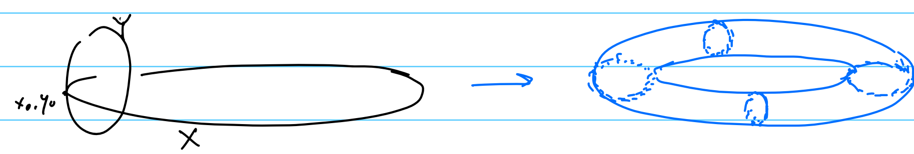
The smash product $X \wedge Y$ is the quotient $X \times Y / X \vee Y$

↳ i.e we are collapsing away the separate factors X and Y.

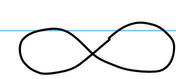
Eg: $S^1 \wedge S^1 = S^2$ —— $S^1 = I / (0 \sim 1)$
$S^m \wedge S^n = S^{m+n}$

Why?

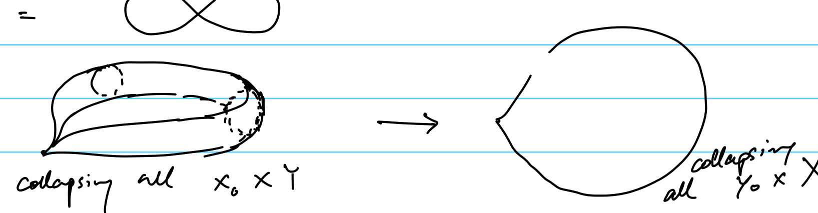
Firstly, $S^1 \times S^1$ results in a torus T^2



Secondly, $S^1 \vee S^1 =$



Then, quotienting:



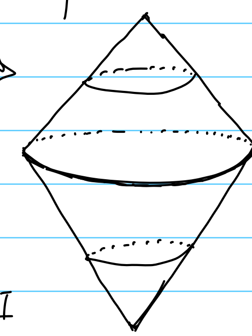
$\boxed{\mathbb{E}}_1$ <u>Suspension</u>

For a space X, the suspension SX is the quotient of $X \times I$ by collapsing $X \times \{0\}$ to a point and $X \times \{1\}$ to another.

<u>Example</u>

(i) $X = S^n$

$SX = S^{n+1}$ with the two suspension points at North and South of S^{n+1}



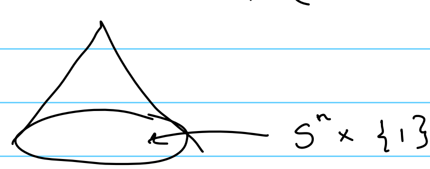
→ We can suspend maps too

$f : X \to Y \rightsquigarrow Sf : SX \to SY$ which is the quotient map of

$f \times \mathbb{1} : X \times I \to Y \times I$

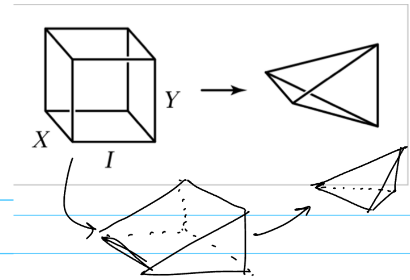
$\boxed{\mathbb{E}}$ <u>Cone</u>:

$$CX = (X \times I) / (X \times \{0\})$$



→ If X is a CW complex, then so are SX and CX as quotients of $X \times I$ with its product cell structure with I given the standard cell structure of two 0-cells joined by one 1-cell.

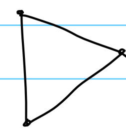
Join



Given X and Y, we can define the space of all line segments joining points in X to points in Y.

$$X * Y = (X \times Y \times I) \Big/ \left(\begin{array}{ll} (x, y_1, 0) \sim (x, y_2, 0) & \forall\, x, x_1, x_2 \in X \\ (x_1, y, 1) \sim (x_2, y, 1) & \forall\, y, y_1, y_2 \in Y \end{array} \right)$$

$\rightarrow$ $pt * pt \longrightarrow$

$pt * pt * pt \longrightarrow$ 

$$\underbrace{pt * pt * \cdots * pt}_{n+1 \text{ points}} = \Delta^n \longrightarrow n\text{-simplex}$$

Reduced Suspension:

$X \to$ CW complex
$\{x_0\} \to$ base point

$$SX = (X \times I) / (X \times \{0\}) \cup (X \times \{1\})$$

$$\Sigma X = SX / (\{x_0\} \times I)$$

Th

Criteria for Homotopy Equivalence

Recall:

Def: Homotopy Equivalence

A map $f : X \to Y$ is a homotopy equivalence if $\exists\, g : Y \to X$ s.t. $f \circ g \simeq id_Y$ and $g \circ f \simeq id_X$

We say the spaces X and Y are homotopy equivalent and $\boxed{X \simeq Y}$

→ can prove easily that this is an equivalence relation.

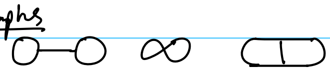
Collapsing Subspaces

Theorem:

If (X, A) is a CW pair consisting of a CW complex X and a contractible subcomplex A, then the quotient map $X \mapsto X/A$ is a homotopy equivalence.

Example

(1) Graphs

(a) 

A B C

→ they are homotopy equivalent

→ collapsing the middle edge of A and C produces B

(b) Let X be a graph with finitely many vertices and edges.

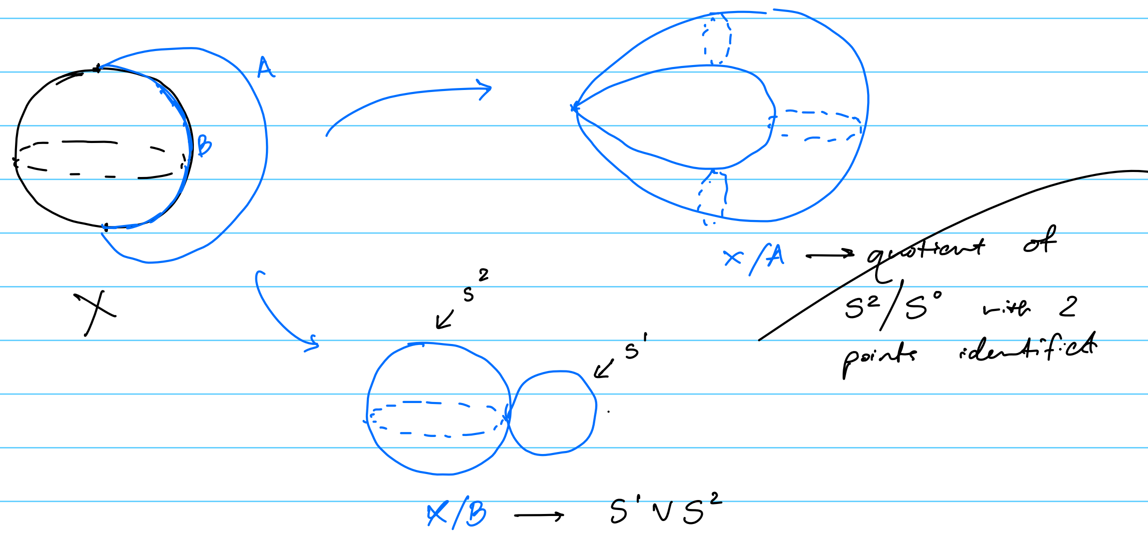
→ if the two endpoints of any edge are distinct, we can collapse it to a pt.

↓

Leads to a homotopy equivalent graph with one less edge.

Can Repeat until all edges are loops.

(2) $X \longrightarrow S^2$ but attach 2 ends of an arc A to N and S pole



Reduced Suspension

$$\Sigma X \simeq SX$$

Attaching Spaces

Start with space X_0 and another space X_1 which we will attach to X_0 by identifying points in a subspace $A \subset X_1$ with points of X_0.

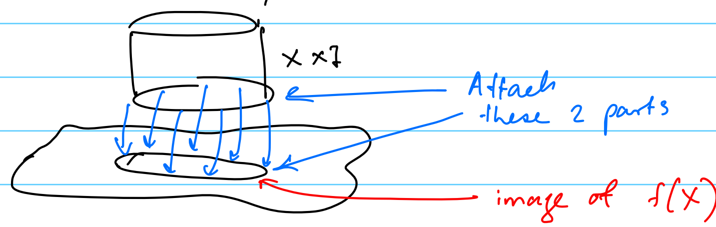
We do this using a map $f : A \longrightarrow X_0$ and then forming a quotient space of $X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A)$

We denote

$$X_0 \sqcup_f X_1 = X_0 \sqcup X_1 / (a \sim f(a), \forall a \in A) \quad \text{where } f : A \longrightarrow X_0,\ A \subset X_1$$

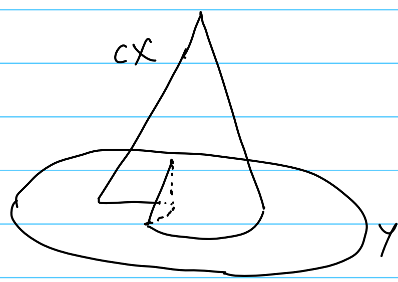
Example:

(1) Mapping cylinder of a map $f : X \longmapsto Y$ is $M_f \to$ the space obtained from Y by attaching $X \times I$ along $X \times \{1\}$ via f.



(2) $\boxed{\text{Mapping Cone}} \longrightarrow$ $C_f = Y \sqcup_f CX$ where $\cancel{C_f}$ CX is the cone $(X \times I)/(X \times \{0\})$

and we attach this to Y along $X \times \{1\}$ via $(x, 1) \sim f(x)$



Example: $X = S^{n-1}$

$C_f \longrightarrow$ attaching to Y the n-cell via $f : S^{n-1} \longmapsto Y$

Proposition

If (X_1, A) is a CW pair and the two attaching maps $f, g : A \longmapsto X_0$ are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1$$

Homotopy Extension Property

Intuition:

Consider the map $f_0 : X \mapsto Y$. Let $A \subset X$ and consider the homotopy on A $f_t : A \mapsto Y$ with $f_0 = f|_A$. We would like to extend this to a homotopy on X as a whole with f_0.

Def: Homotopy Extension

$A \subset X$

(X, A) has the homotopy extension property (h.e.p)

if $\forall Y$, $\forall f_0 : X \longmapsto Y$, $\forall$ homotopy $G : A \times I \longmapsto Y$,

$$G(a, 0) = f_0(a)$$

we can extend G to a homotopy $F : X \times I \longmapsto Y$

$$\text{s.t.} \quad f(x, 0) = f_0(x)$$

$$\Updownarrow$$

$f_0(x) = y$

(X, A) has the h.e.p if every pair of maps $\overbrace{X \times \{0\} \longmapsto Y}$ and $\underbrace{A \times I \longmapsto Y}_{f_t(a)}$ that agree on $\underbrace{A \times \{0\}}_{f_0 = f_0|_A}$ can be extended to a map $\underbrace{X \times I \longmapsto Y}_{f_t : X \longmapsto Y}$.

Lemma:

$$\begin{tikzcd} & X \arrow[dr, dotted] & \\ A \arrow[ur] \arrow[rr, "f"'] & & Y \end{tikzcd}$$

$A \subset X$ top space.

$\forall Y$, any map $f : A \longmapsto Y$ extends to $X \longmapsto Y$ if and only if A is a retract of X

Proof:

$\Leftarrow$ Suppose A is a retract of X via $r : X \longmapsto A$ s.t $r|_A = id_A$

Then $(f \circ r) : X \longmapsto Y$ is our extension

$\Rightarrow$ Suppose, $\forall Y$ and any map $f : A \longmapsto Y$ extends to $X \longmapsto Y$.

i.e $f_t : X \longmapsto Y$ s.t $f_0|_A = f$

Then, let $Y = A$ and $f = id_A$ i.e $id_A : A \longmapsto A$ extends to $f_t : X \longmapsto A$ s.t $f_0|_A = id_A \Rightarrow A$ is a retract of X

Lemma :

A pair (X, A) has the h.e.p if and only if $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

Proof : By hep, the identity map

$\Rightarrow$: $X \times \{0\} \cup A \times I \longmapsto X \times \{0\} \cup A \cup I$ extends to a map

$X \times I \longmapsto X \times \{0\} \cup A \times I$

$\therefore$ $X \times \{0\} \cup A \times I$ is a retract of $X \times I$.

$\Leftarrow$ if A is closed : consider any two maps $X \times \{0\} \longmapsto Y$ and $A \times I \longmapsto Y$ that agree on $A \times \{0\}$. They combine to give a map $X \times \{0\} \cup A \times I \longmapsto Y$ which is continuous by continuity on the closed sets $X \times \{0\}$ and $A \times I$.

Compose this map $X \times \{0\} \cup A \times I \longmapsto Y$ with a retraction $X \times I \longmapsto X \times \{0\} \cup A \times I$ (we have this via hypothesis)

We get an extension $X \times I \longmapsto Y$

$\therefore$ (X, A) has the h.e.p.

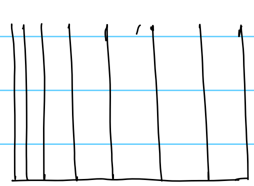
Properties

(i) $\left.\begin{array}{l} \text{H.e.p} \\ X - \text{hausdorff} \end{array}\right\} \Rightarrow A$ is closed in X

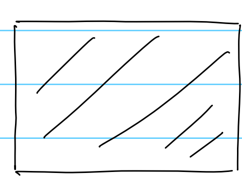
Non-example : (X, A) does not have h.e.p

(1) (I, A) where $A = \{0, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

There is no continuous retraction $I \times I \longmapsto I \times \{0\} \cup A \times I$ because of the structure of (I, A) near 0.



$I \times \{0\} \cup A \times I$



$I \times I$

Consider the ball ~~x_0~~ $B = B(x_0, \frac{1}{2})$

Then. $\exists \delta > 0$ s.t $r(B(x_0, \delta)) \subset B$

γ — ~~path~~ path in B from x_0 to x_1

but x_0 and x_1 are in diff components of $C \cap B$

path at $t=1$

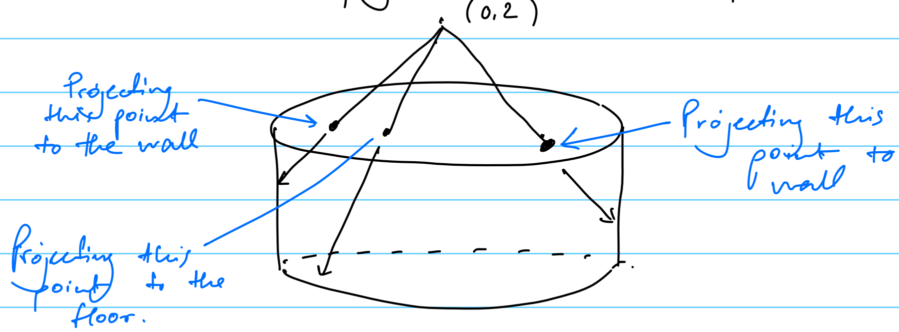
$B(x_0, \delta)$

__Proposition__

If (X, A) is a CW pair, then $X \times \{0\} \cup A \times I$ is a deformation retract of $X \times I$, hence (X, A) has the h.e.p.

__Proof__:

First, note that $\exists$ a retraction $r: D^n \times I \longrightarrow D^n \times \{0\} \cup \partial D^n \times I$ for ex- radial projection from the point $(0,2) \in D^n \times \mathbb{R}$



Now, set $r_t = tr + (1-t)\mathbb{1}$ is a deformation retraction of $D^n \times I$ onto $D^n \times \{0\} \cup \partial D^n \times I$.

Now, with this, we have a deformation retraction of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ since $X^n \times I$ is obtained from $X^n \times \{0\} \cup (X^{n-1} \cup A^n) \times I$ by attaching copies of $D^n \times I$ along $D^n \times \{0\} \cup \partial D^n \times I$.

If we perform the def. ret. of $X^n \times I$ onto $X^n \times \{0\} \cup (X^{n-1} \times A^n) \times I$ during the t-interval $[\frac{1}{2^{n+1}}, \frac{1}{2^n}]$, this infinite concatenation of homotopies is a def. ret. of $X \times I$ onto $X \times \{0\} \cup A \times I$.

Proposition

If the pair (X, A) satisfies h.e.p and A is contractible, then the quotient map $q : X \to X/A$ is a homotopy equivalence.

Proof :

Let $f_t : X \to X$ be the homotopy extending a contraction of A with $f_0 = \mathbb{1}$.

Now, $f_t(A) \subset A$ $\forall t$, so the composition

$$q \circ f_t : X \to X/A$$

sends A to a point and so factors as a composition

$$X \xrightarrow{q} \underbrace{X/A \longrightarrow X/A}_{}$$

Denote this by $\overline{f_t} : X/A \longrightarrow X/A$)

So, $q f_t = \overline{f_t}\, q$

$$\begin{tikzcd}
X \arrow[r, "f_t"] \arrow[d, "q"'] & X \arrow[d, "q"] \\
X/A \arrow[r, "\overline{f_t}"'] & X/A
\end{tikzcd}$$

When $t = 1$, $f_1(A)$ equals to a point (since f_t is homotopy extension of the contraction of A), so f_1 induces a map $g : X/A) \longrightarrow X$ with $gq = f_1$

$$\begin{tikzcd}
X \arrow[r, "f_1"] \arrow[d, "q"'] & X \arrow[d, "q"] \\
X/A \arrow[r, "\overline{f_1}"'] \arrow[ur, "g"] & X/A
\end{tikzcd}$$

So, $qg = \overline{f_1}$ since
$$\begin{aligned}
qg(\overline{x}) &= qgq(x) \\
&= qf_1(x) \\
&= \overline{f_1}\, q(x) \\
&= \overline{f_1}(\overline{x})
\end{aligned}$$

The maps g and q are inverse homotopy equivalences as

$gq = f_1 \simeq f_0 = \mathbb{1}$ via f_t and
$qg = \bar{f}_1 \simeq \bar{f}_0 = \mathbb{1}$ via $\bar{f}_t$.

Def: $W \simeq Z$ rel Y

for (W, Y) and (Z, Y), there are maps $\varphi : W \mapsto Z$ and $\psi : Z \mapsto W$ restricting to identity on Y s.t. $\psi\varphi \simeq \mathbb{1}_W$ and $\varphi\psi \simeq \mathbb{1}_Z$ via homotopies that restrict to the identity on Y at all times.

Proposition

If (X_1, A) is a CW pair and we have attaching maps $f, g : A \mapsto X_0$ that are homotopic, then

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

Proof:

Let $F : A \times I \mapsto X_0$ is a homotopy from f to g, consider the space $X_0 \sqcup_F (X_1 \times I)$ which have both $X_0 \sqcup_f X_1$ and $X_0 \sqcup_g X_1$ as subspaces

We can deformation retract $X_1 \times I$ onto $X_1 \times \{0\} \cup A \times I$ which induces a def refraction of

$$X_0 \sqcup_F (X_1 \times I) \text{ onto } X_0 \sqcup_f X_1$$

Similarly, $X_0 \sqcup_F (X_1 \times I)$ def retracts onto $X_0 \sqcup_g X_1$

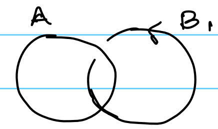
Both of them are identity on X_0 so we get the homotopy equivalence

$$X_0 \sqcup_f X_1 \simeq X_0 \sqcup_g X_1 \text{ rel } X_0$$

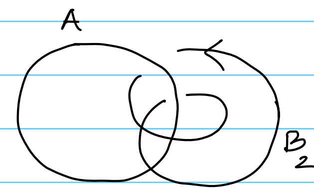
Fundamental Group

Intuition

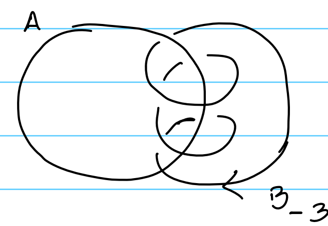
Two linked circles in $\mathbb{R}^3$:



Link B with A two times in the forward direction :



Link B with A three times in the backward direction :

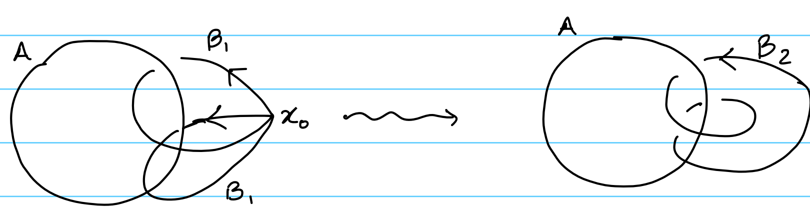


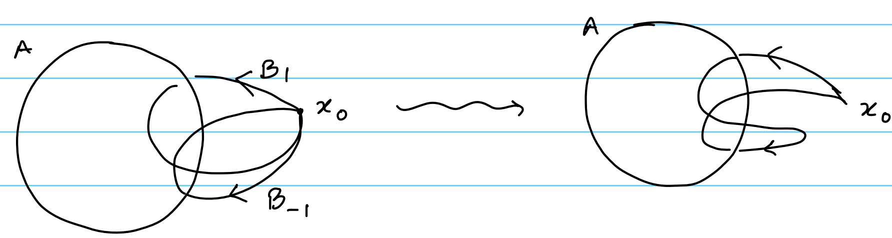
B_2 and B_{-3} are oriented circles/loops.

Two loops, B and B', starting and ending at the same point x_0 can be added to form a new loop that travels around B and B'.

So, $B_1 + B_1 = B_2$

$B_1 + B_{-1} = B_0$ ← unlinked from A





More generally, $B_m + B_n = B_{m+n}$

Paths and Homotopy of paths

Def: Path in X

A continuous map $f: I \longmapsto X$ where $I = [0,1]$

Def: Homotopy of paths

A family $f_t : I \longrightarrow X$ where $t \in I$ s.t

(1) $f_t(0) = x_0$ and $f_t(1) = x_1 \; \forall t$

(2) the associated map $F: I \times I \longmapsto X$ is continuous

We say $f_0 \simeq f_1$.

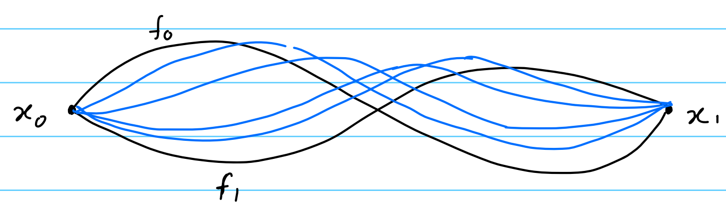
$\rightarrow$ $f_0 \simeq f_1$ means homotopic rel. $\partial I = \{0,1\}$ ie the endpoints are fixed.

Examples

(1) Linear homotopies in $\mathbb{R}^n$:

Any 2 paths f_0 and f_1 in $\mathbb{R}^n$ with endpoints ~~in~~ x_0 and x_1 are homotopic by $f_t(x) = (1-t) f_0(x) + t f_1(x)$

Here, $F(x,t) = f_t(x) = (1-t) f_0(x) + t f_1(x)$ is continuous since f_0 and f_1 are continuous. and sum ~~&~~ and scalar multiplication preserve continuity.



Non-example

$f_0, f_1 : I \longrightarrow S^1$

$\left.\begin{array}{l} f_0(t) = 1 \\ f_1(t) = e^{2\pi i t} \end{array}\right\}$ they are not path homotopic

Proposition

The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation.
We denote the equivalence class of f by $[f]$ and is called the homotopy class of f.

Proof:

Reflexivity: $f \simeq f$ by homotopy $f_t = f$

Symmetry: If $f_0 \simeq f_1$ via f_t, then $f_1 \simeq f_0$ via f_{1-t}.

Transitivity: Suppose $f_0 \simeq f_1$ via f_t. and if $f_1 = g_0$ with $g_0 \simeq g_1$ via g_t, then the homotopy

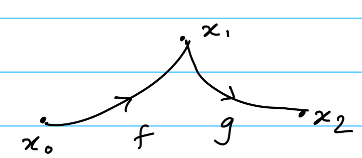
$$h_t = \begin{cases} f_{2t}, & t \in [0, \frac{1}{2}] \\ g_{2t-1}, & t \in [\frac{1}{2}, 1] \end{cases}$$

The associated function $H(s,t) = h_t(s)$ is continuous.
A function on the union of 2 closed sets is continuous if it is continuous restricted to each of the 2 sets separately.

Def: Product Path

Given two paths $f, g : I \longmapsto X$ s.t $f(1) = g(0)$, the product path $f \cdot g$ first traverses f and then g:

$$f \cdot g(t) = \begin{cases} f(2t), & 0 \le t \le \frac{1}{2} \\ g(2t-1), & \frac{1}{2} \le t \le 1 \end{cases}$$



This product path preserves homotopy classes:
if $f_0 \simeq f_1$ and $g_0 \simeq g_1$ via f_t and g_t homotopies respectively and if $f_0(1) = g_0(0)$ so that $f_0 g_0$ is well-defined then $f_t \cdot g_t$ provides the homotopy

$$f_0 g_0 \simeq f_1 g_1$$

Def : Loop

Paths $f : I \to X$ s.t $f(0) = f(1) = x_0 \in X$

$x_0 \longrightarrow$ basepoint

$\rightarrow$ The set of all homotopy classes $[f]$ of loops $f : I \to X$ at the basepoint $x_0 \in X$ is denoted $\pi_1(X, x_0)$

Proposition :

$\pi_1(X, x_0)$ is a group w.r.t the product
$$[f][g] = [f \cdot g]$$
This group is called the fundamental group of X at basepoint x_0.

Proof :

Since the basepoint $x_0 \in X$ is fixed, the product of any two paths, f and g in $\pi_1(X, x_0)$ is defined.

Firstly, define reparametrisation of a path f to be a composition $f\varphi$ where $\varphi : I \to X$ is a continuous map s.t $\varphi(0) = 0$ and $\varphi(1) = 1$.

Reparametrisation preserves homotopy class of f since $f\varphi \simeq f$ via homotopy $f\varphi_t$ where $\varphi_t(x) = (1-t)\varphi(x) + tx$

so $\varphi_0(x) = \varphi(x)$ and $\varphi_1(x) = x$

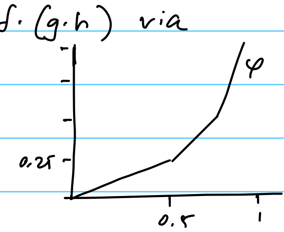
We often show that f is a reparametrization of g to prove $f \simeq g$.

Given the paths f, g and h with $f(1) = g(0)$ and $g(1) = h(0)$, then both $(f \cdot g) \cdot h$ and $f \cdot (g \cdot h)$ are defined.

Note $(f \cdot g) \cdot h$ is a reparametrisation of $f \cdot (g \cdot h)$ via $f \cdot (g \cdot h) = (f \cdot g) \cdot h\,\varphi$ where φ is a continuous map s.t

$\varphi : [0, \frac{1}{2}] \longmapsto [0, \frac{1}{4}]$

$\varphi : [\frac{1}{2}, 1] \longmapsto [\frac{1}{4}, 1]$



So, $(f \cdot g) \cdot h \simeq f \cdot (g \cdot h)$

Given a path $f: I \mapsto X$, let c be the constant path at $f(1)$ defined by $c(s) = f(1)$, $\forall s \in I$.
Then, $f \cdot c$ is a reparametrisation of f:

$$f \cdot c(x) = \begin{cases} f(2x) & \text{for } x \in [0, \frac{1}{2}] \\ c(2x-1) & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$$

So, $f \cdot c = f\varphi$ where $\varphi: [0, \frac{1}{2}] \longrightarrow [0, 1]$
$\varphi: [\frac{1}{2}, 1] \longrightarrow 1$

$\therefore f \cdot c \simeq f$
Similarly $c \cdot f \simeq f$ where c is constant path at $f(0)$.

Taking f to be a loop, the homotopy class of the constant path is a two-sided identity.

Now, let f be a path from x_0 to x_1. Its inverse path is $\bar{f}$ from x_1 to x_0. defined by $\bar{f}(s) = f(1-s)$
Then, $f \cdot \bar{f}$ is homotopic to a constant path via homotopy
$h_t = f_t \cdot g_t$
where $f_t = f$ on $[0, 1-t]$ and $f_t = f(1-t)$ on $[1-t, 1]$
and $g_t = \bar{f}_t$

Then, $f_0 = f$ and $f_1 =$ constant path c at x_0
So, h_t is a homotopy from $f \cdot \bar{f}$ to $c \cdot \bar{c}$
as $h_0 = f_0 g_0 = \begin{cases} f & \text{for } x \in [0, \frac{1}{2}] \\ \bar{f} & \text{for } x \in [\frac{1}{2}, 1] \end{cases}$

$$h_1 = f_1 \cdot g_1 = \begin{cases} c, & x \in [0, \frac{1}{2}] \\ \bar{c}, & x \in [\frac{1}{2}, 1] \end{cases}$$

$\therefore f \cdot \bar{f} \simeq c$ (defining $c \cdot \bar{c} = c$) where $c = x_0$
Replacing f by $\bar{f}$ gives $\bar{f} \cdot f = c$
Take f to be the loop at x_0, then $[\bar{f}]$ is a 2-sided inverse for $[f]$ in $\pi_1(X, x_0)$.

Fundamental Group of X at x_0 : $\pi_1(X, x_0)$

$$\pi_1(X, x_0) = \{\text{loops from } x_0 \text{ to itself in } X\} / (\text{path homotopy})$$

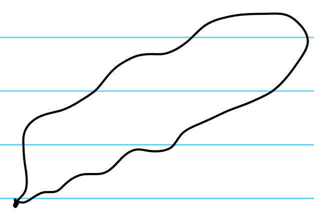
$\longrightarrow$ $[f]\cdot[g] = [fg]$

$\longrightarrow$ $[f]^{-1} = [\bar{f}]$ where $\bar{f}(t) = f(1-t)$

$\longrightarrow$ $[\text{constant}_{x_0}] = 1$

Examples

(1) $\pi_1(\mathbb{R}^n, 0) = 1$



$f : I \to \mathbb{R}^n$

We can contract it to be constant at 0 by $f_t(x) = tx$

We say $\pi_1(X) = 1$ if X is contractible

$f_t(x) = r_t \circ f$

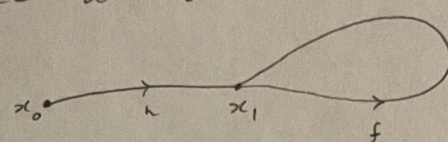
homotopy from id_X to constant map

Cts

<u>Change of basepoint</u>

Let x_0 and x_1 lie in the same path-component of X.
Let $h : I \longrightarrow X$ be a path from x_0 to x_1 with the inverse path $\bar{h}(s) = h(1-s)$. from x_1 to x_0.

Then, for each loop f based at x_1, define the loop $h \cdot f \cdot \bar{h}$ based at x_0.



Alternatively, we can define a general n-fold product $f_1 \cdots\cdots f_n$ in which the path f_i is traversed in $\left[\frac{i-1}{n}, \frac{i}{n}\right]$.

Then define the <u>change of basepoint map</u> $\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ by $\beta_h[f] = [h \cdot f \cdot \bar{h}]$

<u>Proposition</u>: The map $\beta_h : \pi_1(X, x_1) \longrightarrow \pi_1(X, x_0)$ is an isomorphism.
So, $\pi_1(X, x_0) \cong \pi_1(X, x_1)$

<u>Proof</u>: homomorphism as

$$\beta_h[f \cdot g] = [h \cdot f \cdot g \cdot \bar{h}] = [h \cdot f \cdot \bar{h} \cdot h \cdot g \cdot \bar{h}]$$
$$= \beta_h[f]\,\beta_h[g]$$

This has the inverse $\beta_{\bar{h}}$ as

$$\beta_h \beta_{\bar{h}}[f] = \beta_h[\bar{h} \cdot f \cdot h] = [h \cdot \bar{h} \cdot f \cdot h \cdot \bar{h}]$$
$$= [f]$$

Similarly, $\beta_{\bar{h}} \beta_h[f] = [f]$

<u>Def : Simply connected</u>

A space is <u>simply connected</u> if it is $\boxed{\text{path connected}}$ and has $\boxed{\text{trivial fundamental group.}}$

ie the constant path

Proposition

A space X is simply connected iff there is a unique homotopy class of paths connecting any two points in X.

Proof :

$\Rightarrow$: Need to show uniqueness.

Suppose

Let f and g be 2 paths from x_0 to x_1

Then $f \simeq f \cdot \bar{g} \cdot g \simeq g$ since the loops $\bar{g} \cdot g$ and $f \cdot \bar{g}$ are each homotopic to constant loops, given $\pi_1(X) = 0$

$\Leftarrow$: If there is only one homotopy class of ~~paths~~ loops at x_0, then all loop at x_0 are homotopic to the constant loop

$\therefore \pi_1(X, x_0) = \pi_1(X) = 0$

If X is path connected, then $\pi_1(X, x_0)$ is independent of x_0. We write it as $\pi_1(X)$.

Induced Homomorphism

Def: Induced Homomorphism

Suppose, $\varphi : X \to Y$ is a map taking basepoint $x_0 \in X$ to the basepoint $y_0 \in Y$

We say $\varphi : (X, x_0) \mapsto (Y, y_0)$

Then, φ induces a homomorphism

$$\varphi_* : \pi_1(X, x_0) \mapsto \pi_1(Y, y_0)$$

defined by composing the loops $f : I \to X$ based at x_0 with φ :

$$\varphi_*([f]) = [\varphi f]$$

→ Well-defined :

Homotopy f_t of loops at x_0 yields a homotopy φf_t of loops based at y_0

$$\therefore \quad \varphi_*([f_0]) = [\varphi f_0] = [\varphi f_1] = \varphi_*[f_1]$$

→ φ_* is a homomorphism :

$$\varphi_*(f \cdot g) = \varphi(f \cdot g)$$
$$= \varphi f \cdot \varphi g$$
$$= \varphi_*(f) \cdot \varphi_*(g)$$

→ both functions have values

$\varphi f(2s)$, $\quad 0 \le s \le \frac{1}{2}$

$\varphi g(2s-1)$, $\quad \frac{1}{2} \le s \le 1$

Properties of induced homomorphisms

(1) $(X, x_0) \xrightarrow{\psi} (Y, y_0) \xrightarrow{\varphi} Z, z_0$

$$(\varphi\psi)_* = \varphi_* \psi_* : \pi_1(X, x_0) \longmapsto \pi_1(Z, z_0)$$

Proof :

$$(\varphi\psi) f = \varphi(\psi f)$$

(2) $\mathbb{1}_* = \mathbb{1}$ which is saying $\mathbb{1} : X \to X$ induces $\mathbb{1} : \pi_1(X, x_0) \mapsto \pi_1(X, x_0)$

(3) If φ is a homeomorphism with inverse φ^{-1} then φ_* is an ~~k~~ isomorphism with inverse $(\varphi^{-1})_*$ since

$$\varphi_* \varphi^{-1}_* = (\varphi \varphi^{-1})_* = \mathbb{1}_* = \mathbb{1} \quad \text{and similarly} \quad \varphi^{-1}_* \varphi_* = \mathbb{1}$$

(4) Let $\varphi, \psi : X \longrightarrow Y$.
If φ and ψ are homotopic, then $\psi_* = \varphi_*$

Proof : $\varphi_*[f] = [\varphi f]$
$= [\psi f]$ (via homotopy of φ and ψ)
$= \psi_*[f]$

(5) Proposition :

If a space X retracts onto a subspace A, then the induced homomorphism $i_* : \pi_1(A, x_0) \longmapsto \pi_1(X, x_0)$ induced by the inclusion $i : A \hookrightarrow X$ is injective. If A is a deformation retract of X, then i_* is an isomorphism

Proof :

Suppose, X retracts onto $A \subset X$. via $r : X \longrightarrow A$
Then $ri = id_A$
So, $(ri)_* = r_* i_* = \mathbb{1}$

Therefore i_* is injective

Suppose $i_*(f) = 1$ for some $f \in \pi_1(A, x_0)$
Then, $(r_* i_*)(f) = r_*(1) = 1$. But $r_* i_* = \mathbb{1}$
so $f = 1$.

Now, suppose X def. retracts onto A via $r_t : X \mapsto X$
So, $r_0 = id_X$, $r_t|_A = id_A$ and $r_1(X) \subset A$
then, for any loop $f : I \longrightarrow X$ based at $x_0 \in A$,
the composition $r_t f$ gives a homotopy of f to a loop in A, so i_* is also surjective.

as $r_1(X) \subseteq A$

i.e for any $f : I \to X$,
first def retract to $f' : I \to A$
where $f' = r_1 of$. Then $i_*(f') = f' \in \pi_1(X, x_0)$
and $[f'] = [f]$ by the homotopy r_t.

Lemma 1.15

If a space X is the union of a collection of path connected open sets A_α, each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

Proof:

Consider a loop $f: I \longrightarrow X$ at x_0.

Partition I into $0 = s_1 < s_2 < \dots < s_m = 1$ s.t each subinterval $[s_{i-1}, s_i]$ is mapped by f to a single A_α

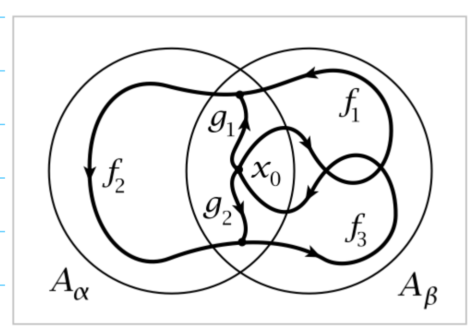
$\hookrightarrow$ Since f is continuous, each s in I has an open nbhd $V_s \subset I$ s.t f maps V_s to ~~A_α~~ some A_α. We can take $V_s \subset I$ s.t f maps $\overline{V_s}$ (closure of V_s) to a single V_α
The enpoints of this finite set of intervals will define the portition $0 = s_1 < s_1 \dots < s_m = 1$

We denote A_i to be the set containing $f([s_{i-1}, s_i])$ and we let f_i be the path obtained by restricting $f|_{[s_{i-1}, s_i]}$

Now, f is the composition $f_1 \cdots f_m$ with f_i, a path in A_i.
Since $A_i \cap A_{i+1}$ is path connected, we can find a path $g_i \in A_i \cap A_{i+1}$ from x_0 to the point $f(s_i) \in A_i \cap A_{i+1}$.
Then, the loop

$$(f_1 \cdot \overline{g_1}) \cdot (\overline{g_1} \cdot f_2 \cdot \overline{g_2}) \cdot \dots \dots (g_{m-1} \cdot f_m)$$

is homotopic to f and is a composition of loops that each lie in a single A_i.



Def: Basepoint Preserving Homotopy

Consider a homotopy φ_t taking $A \subset X$ to a subspace $B \subset Y$ for all t, then we speak of maps of pairs

$$\varphi_t : (X, A) \longmapsto (Y, B)$$

A basepoint-preserving homotopy $\varphi_t : (X, x_0) \longmapsto (Y, y_0)$ is the case where $\varphi_t(x_0) = y_0 \quad \forall t$.

(6) If $\varphi_t : (X, x_0) \longmapsto (Y, y_0)$ is a basepoint preserving homotopy, then $\varphi_{0*} = \varphi_{1*}$

Proof:
$$\begin{aligned} \varphi_{0*}[f] &= [\varphi_0 f] \\ &= [\varphi_1 f] \quad \text{(via homotopy } \varphi_t f\text{)} \\ &= \varphi_{1*}[f] \end{aligned}$$

Def: Homotopy Equivalence for spaces with basepoints

We say $(X, x_0) \simeq (Y, y_0)$ if there are maps $\varphi : (X, x_0) \longmapsto (Y, y_0)$ and $\psi : (Y, y_0) \longmapsto (X, x_0)$ with homotopies $\varphi\psi \simeq \mathbb{1}_{(Y, y_0)}$ and $\psi\varphi \simeq \mathbb{1}_{(X, x_0)}$. <u>through maps that fix the basepoint.</u>

In this case, the induced maps on π_1 satisfy

$$\varphi_* \psi_* = (\varphi\psi)_* = \mathbb{1}_* = \mathbb{1}$$

$$\psi_* \varphi_* = (\psi\varphi)_* = \mathbb{1}_* = \mathbb{1}$$

$\therefore$ φ_* and ψ_* are ~~is~~ inverse isomorphisms

$$\therefore \quad \pi_1(X, x_0) \cong \pi_1(Y, y_0)$$

What if φ_t does not ~~also~~ send x_0 to a fixed $y_0 \in Y$ for all t? This means the basepoint in X is not always mapped to the same point by a homotopy.

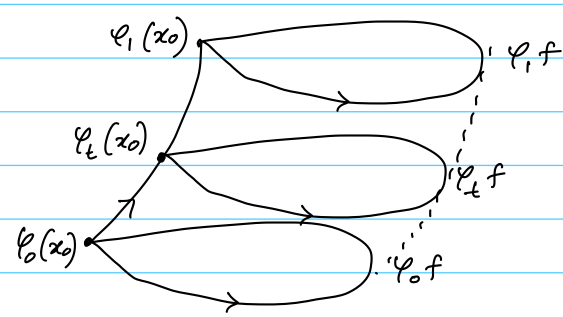
Lemma:

If $\varphi_t : X \mapsto Y$ is a homotopy and h is the path $\varphi_t(x_0)$ formed by the images of a basepoint $x_0 \in X$, then the three maps in the diagram satisfy $\varphi_{0*} = \beta_h \varphi_{1*}$

$$\begin{tikzcd} & \pi_1(Y, \varphi_1(x_0)) \arrow[d, "\beta_h"] \\ \pi_1(X, x_0) \arrow[ur, "\varphi_{1*}"] \arrow[r, "\varphi_{0*}"'] & \pi_1(Y, \varphi_0(x_0)) \end{tikzcd}$$

Proof:

Let h_t be the restriction of h to the interval $[0,t]$ (with a reparametrization so that domain of h_t is $[0,1]$):
So. $h_t(s) = h(ts)$ where $h : I \longmapsto Y$ with $h(\tilde{t}) = \varphi_{\tilde{t}}(x_0)$



Then, if f is a loop in X at basepoint x_0, then the product $h_t \cdot (\varphi_t f) \cdot \bar{h}$ gives a homotopy of loops at $\varphi_0(x_0)$.

Restricting this ~~to~~ to $t = 0$ and $t = 1$, we see $\varphi_{0*}([f]) = \beta_h(\varphi_{1*}([f]))$

$$X \simeq Y \implies \pi_1(X, x_0) \approx \pi_1(Y, \varphi(x_0))$$

Theorem :

If $\varphi : X \longmapsto Y$ is a homotopy equivalence, then the induced homomorphism $\varphi_* : \pi_1(X, x_0) \longmapsto \pi_1(Y, \varphi(x_0))$ is an isomorphism for $\forall x_0 \in X$.

$$\therefore \pi_1(X, x_0) \longrightarrow \pi_1(Y, \varphi(x_0))$$

Proof :

Let $\varphi : X \longmapsto Y$ be a homotopy equivalence

So, Let $\psi : Y \longmapsto X$ be the homotopy inverse

So, $\varphi\psi \simeq \mathbb{1}$

$\psi\varphi \simeq \mathbb{1}$

$$\pi_1(X, x_0) \xrightarrow{\varphi_*} \pi_1(Y, \varphi(x_0)) \xrightarrow{\psi_*} \pi_1(X, \psi\varphi(x_0)) \xrightarrow{\varphi_*} \pi_1(Y, \varphi\psi\varphi(x_0))$$

Given $\psi\varphi \simeq \mathbb{1}$, then $\psi_*\varphi_* = \beta_h$ for some h by the previous Lemma. $\implies \psi_*\varphi_*$ is an isomorphism

Since $\psi_*\varphi_*$ is an isomorphism

φ_* is injective.

Similarly, with $\psi_*\varphi_*$, we conclude ψ_* is injective.

$\therefore$ φ_*, ψ_* are injections and $\psi_*\varphi_*$ is an isomorphism. so φ_* is a surjection too.

(4) $\pi_1(X \times Y, (x_0, y_0)) = \pi_1(X, x_0) \times \pi_1(Y, y_0)$

Proof :

A path $I \longrightarrow X \times Y$

is a pair of paths $(f : I \to X, g : I \to Y)$

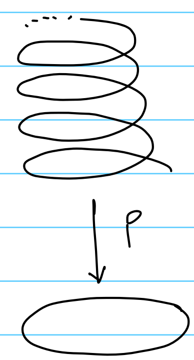
Fundamental Group of the Circle

Some preliminary tools:

(1) Let $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ for $s \in I$ be a loop based at $(1,0)$.
Then, $[\omega]^n = [\omega_n]$ where $\omega_n(s) = (\cos 2\pi n s, \sin 2\pi n s)$ for $n \in \mathbb{Z}$.
by the definition of product path and the fact that product preserves homotopy.

(2) Compare paths in S^1 with paths in $\mathbb{R}$:
$\rightarrow$ Let $p : \mathbb{R} \longmapsto S^1$ via $p(s) = (\cos 2\pi s, \sin 2\pi s)$

Visualization: first, consider the helix $s \longmapsto (\cos 2\pi s, \sin 2\pi s, s)$
Then, project $\mathbb{R}^3$ onto $\mathbb{R}^2$ by $(x, y, z) \longmapsto (x, y)$
So, projecting the helix onto $\mathbb{R}^2$ gives p



$\rightarrow$ $\boxed{\omega_n(s) = p\tilde{\omega}_n(s)}$ where $\underbrace{\tilde{\omega}_n : I \longmapsto \mathbb{R} \text{ is the path } \tilde{\omega}_n(s) = ns}_{\text{starts at } 0 \text{ and ends at } n}$

$\tilde{\omega}_n$ is called the lift of ω_n.

$p\tilde{\omega}_n(s)$ winds around the helix $|n|$ times $\rightarrow$ upwards if $n > 0$ and downwards if $n < 0$.

(3) <u>Def: Covering Space</u>

Given a space X, a covering space of X consists of a space $\tilde{X}$ and a map $p : \tilde{X} \longmapsto X$ satisfying:

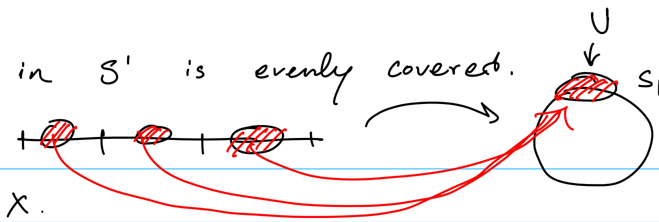
(a) For each $x \in X$, $\exists$ open neighbourhood $U \ni x$ in X s.t
$p^{-1}(U) = \coprod_{\alpha \in A} V_\alpha$ where each V_α is open and each <u>V_α is mapped homeomorphically onto U by p.</u>
is a union of disjoint open sets (each of

We say U is <u>evenly covered</u>.

$p|_{V_\alpha} : V_\alpha \longrightarrow U$ is a homeomorphism

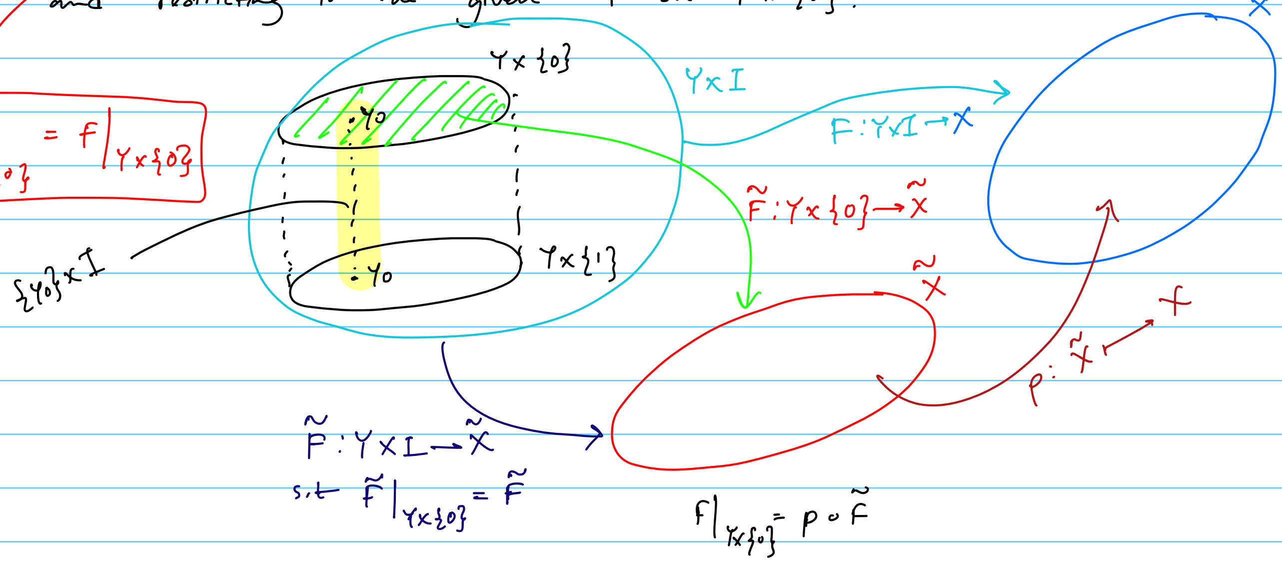
Example:

(1) $p: \mathbb{R} \longmapsto S^1$, an open arc in S^1 is evenly covered.
Define it by $p(\theta) = e^{2\pi i \theta}$.



Lemma: Consider covering spaces $p: \tilde{X} \mapsto X$.
Given a map $F: Y \times I \to X$ and a map $\tilde{F}: Y \times \{0\} \longmapsto \tilde{X}$ lifting $F|_{Y \times \{0\}}$, then there is a unique map $\tilde{F}: Y \times I \to \tilde{X}$ lifting F and restricting to the given $\tilde{F}$ on $Y \times \{0\}$.

$$\boxed{\therefore\ p \circ \tilde{F}|_{Y \times \{0\}} = F|_{Y \times \{0\}}}$$



Proof:

First, construct a lift $\tilde{F}: N \times I \to \tilde{X}$ for some neighbourhood N of $y_0 \in Y$.

$\hookrightarrow$ Given F is continuous, $\forall\ (y_0, t) \in Y \times I$ has a product neighbourhood $N_t \times (a_t, b_t)$ s.t $F(N_t \times (a_t, b_t))$ is contained in an evenly covered neighbourhood of $F(y_0, t)$.

$\longrightarrow$ around $F(y_0, t)$, $\exists$ an evenly covered neighbourhood, since $p: \tilde{X} \longmapsto X$ is a covering space. By continuity, we can always shrink $N_t \times (a_t, b_t)$ so that $F(N_t \times (a_t, b_t))$ is inside this evenly covered nbd.

By compactness of $\{y_0\} \times I$, finitely many such $N_t \times (a_t, b_t)$ products cover $\{y_0\} \times I$. Thus, we can choose one neighbourhood N of $\{y_0\}$ and a partition $0 = t_0 < t_1 < \dots < t_m = 1$ of I s.t for each i, $F(N \times [t_i, t_{i+1}])$ is contained in an evenly covered neighbourhood U_i.

Assume, inductively, $\tilde{F}$ has been constructed on $N \times [0, t_i]$ starting with our given $\tilde{F}$ on $N \times \{0\}$. Thus, $F(N \times [t_i, t_{i+1}]) \subset U_i$, so since U_i is evenly covered, $\exists$ open set $\tilde{U}_i \subset \tilde{X}$ projecting homeomorphically onto U_i by p and containing $\tilde{F}(y_0, t_i)$ because

$\tilde{F}|_{N\times[0,t_i]}$ is a lift of $F|_{N\times[0,t_i]}$ so $p(\tilde{F}(y_0,t_i)) = F(y_0,t_i)$ - we know it is a lift as we have already constructed the lift on $N\times[0,t_i]$.

We can extend $\tilde{F}$ on $N\times[t_i,t_{i+1}]$ by composing $p^{-1}: U_i \to \tilde{U}_i$ (since p is a homeomorphism) with F. For this to be continuous, this must agree with $\tilde{F}$ on $N\times[0,t_i]$, in particular at (y_0,t_i).

Replace N by a small enough nbd of y_0, we get that $\tilde{F}(N\times\{t_i\})$ is contained in $\tilde{U}_i \Rightarrow$ by replacing $N\times\{t_i\}$ by its intersection with $\left(\tilde{F}|_{N\times\{t_i\}}\right)^{-1}(\tilde{U}_i)$. Then define $\tilde{F}$ on $N\times[t_i,t_{i+1}]$ to be the composition of F with $p^{-1}: U_i \to \tilde{U}_i$.

Continuing, we get $\tilde{F}: N\times I \to \tilde{X}$, a lift, for some neighbourhood N of y_0.

Next we show uniqueness of this lift. We prove for when Y is a point. Since Y is a point, we suppress it from our notation.

Let $\tilde{F}$ and $\tilde{F}'$ be 2 lifts of $F: I \to X$ s.t $\tilde{F}(0) = \tilde{F}'(0)$.

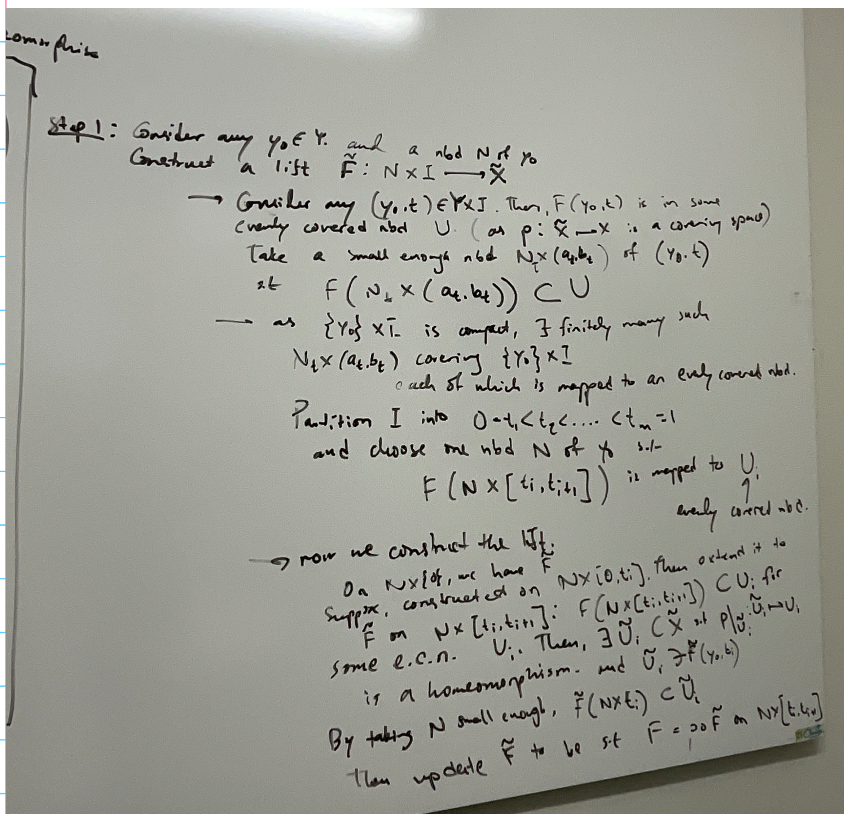
Again, we choose a partition $0 = t_1 < t_2 < \dots < t_m = 1$ of I s.t for each i, $F([t_i,t_{i+1}])$ is contained in evenly covered nbd U_i.

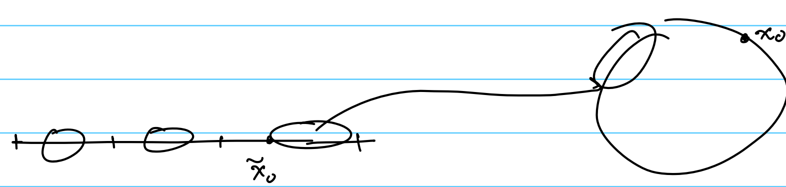
Assume inductively that $\tilde{F} = \tilde{F}'$ on $[0,t_i]$. As $[t_i,t_{i+1}]$ is connected, so is $\tilde{F}([t_i,t_{i+1}]) \Rightarrow$ it must lie in single one of the disjoint open sets $\tilde{U}_i$ projecting homeomorphically to U_i. By same logic, $\tilde{F}'([t_i,t_{i+1}])$ lies in a single $\tilde{U}_i$ and it must be the same one as $\tilde{F}'(t_i) = \tilde{F}(t_i)$. As p is injective on $\tilde{U}_i$ and $p\tilde{F} = p\tilde{F}'$, we get $\tilde{F} = \tilde{F}'$ on $[t_i,t_{i+1}]$. Continuing this way, $\tilde{F} = \tilde{F}'$.

Lastly, observe that since the lift $\tilde{F}$ constructed on sets of the form $N\times I$ is unique when restricted to each segment $\{y\}\times I$, they must agree when two such sets $N\times I$ overlap.

$\therefore$ we have a well-defined lift $\tilde{F}$ on all of $Y \times I$.

$\tilde{F}$ is continuous as it is continuous on each $N \times I$. and unique as it is unique on each segment $\{y\} \times I$.





Path Lifting Property

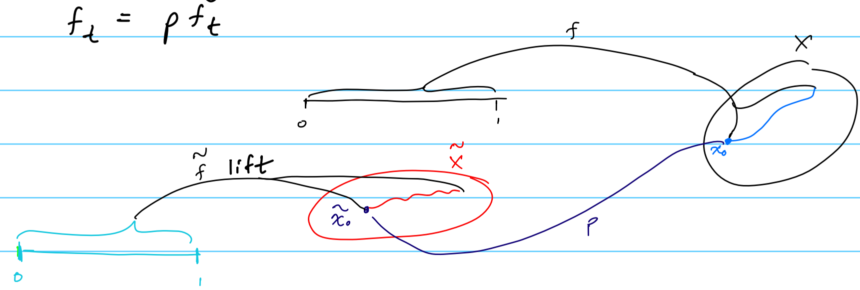
!! **Lemma**: Consider covering spaces $p: \tilde{X} \mapsto X$.

(1) For each path $f: I \to X$ s.t $f(0) = x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lift $\tilde{f}: I \to \tilde{X}$ starting at $\tilde{x}_0$.

Hence, $f = p\tilde{f}$.

(2) For each homotopy $f_t: I \to X$ of paths s.t $f_t(0) = x_0$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lifted homotopy $\tilde{f}_t: I \to \tilde{X}$ of paths starting at $\tilde{x}_0$.

Hence, $f_t = p\tilde{f}_t$



$\tilde{f}: I \longmapsto \tilde{X}$

$p\tilde{f}: I \longmapsto X$, $f: I \longmapsto X$

$\tilde{f}(0) = \tilde{x}_0$ $\quad$ $f(0) = x_0$

$p\tilde{f}(0) = x_0$

$f = p\tilde{f}$

Proof:

(1) follows from prev. lemma when Y is a point

(2) Let $Y = I$.

Then for the homotopy $f_t: I \to X$, we have a map $F: I \times I \mapsto X$ with $F(s,t) = f_t(s)$.

We get a unique lift $\tilde{F}: I \times \{0\} \mapsto \tilde{X}$ using part (1).

Then, by prev. lemma, we get a unique lift $\tilde{F}: I \times I \to \tilde{X}$.

The restrictions $\tilde{F}|_{\{0\} \times I}$ and $\tilde{F}|_{\{1\} \times I}$ are paths lifting constant paths, so they must also be constant by uniqueness of part (1).

So. $\tilde{f}_t(s) = \tilde{F}(s,t)$ is a homotopy of paths and $\tilde{f}_t$ lifts f_t as $F = p\tilde{F}$

We set $\tilde{X}$ to be $\mathbb{R}$ here as $p: \mathbb{R} \to S^1$ is a covering space.

Theorem: $\pi_1(S^1)$ is an infinite cyclic group generated by the homotopy class of the loop $\omega(s) = (\cos 2\pi s, \sin 2\pi s)$ based at $(1,0)$. So, $\pi_1(S^1) \cong \mathbb{Z}$ as a group.

Proof: Let $f: I \longrightarrow S^1$ be a loop at the basepoint $x_0 = (1,0)$ which is one element of the group $\pi_1(S^1, x_0)$.

Then, $\underbrace{\exists \text{ a lift } \tilde{f}}^{\text{by previous lemma (1)}}$ starting at 0 and must end at some integer n since $p\tilde{f}(1) = f(1) = x_0$ and $\underbrace{p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}}_{\text{recall } p(s) = e^{2\pi i s}}$.

Another path in $\mathbb{R}$ from 0 to n is $\underbrace{\tilde{\omega}_n}_{\text{recall}}$ and $\tilde{\omega}_n(s) = ns$

$\tilde{f} \simeq \tilde{\omega}_n$ via the linear homotopy $(1-t)\tilde{f} + t\tilde{\omega}_n$. Compose the homotopy with p gives the homotopy $\tilde{f} \simeq \tilde{\omega}_n$ so $[f] = [\omega_n]$.

$\therefore$ for any loop f, $f = [\omega_n]$. ~~Can~~ Is n fixed here? Yes.

Next, we show that n is uniquely determined by $[f]$: Suppose $f \simeq \omega_n$ and $f \simeq \omega_m$. Let f_t be a homotopy from $\omega_m = f_0$ ~~to~~ to $\omega_n = f_1$.

Then, $\underbrace{\text{this } f_t \text{ lifts}}_{\text{by previous lemma (2)}}$ to a homotopy $\tilde{f}_t$ of paths starting at 0

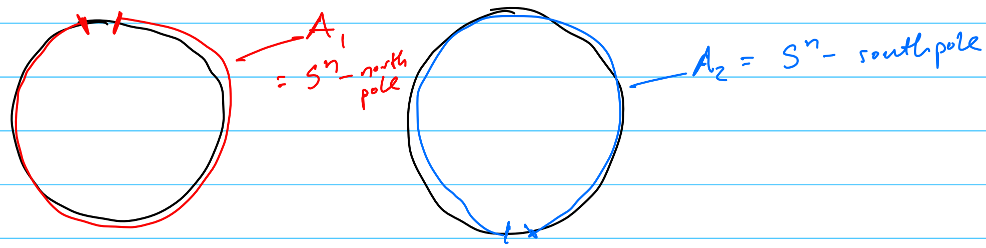
The uniqueness of $\tilde{f}$ (by prev. lemma) implies that $\tilde{f}_0 = \tilde{\omega}_m$ and $\tilde{f}_1 = \tilde{\omega}_n$. Since $\tilde{f}_t$ is a homotopy of paths, the endpoint $\tilde{f}_t(1)$ is independent of t. For $t=0$, the endpoint is m and for $t=1$, it is n. So, $m = n$.

The fact that this group is generated by $\omega(s)$ is obvious from noting that $[\omega]^n = [\omega_n]$

Proposition: $\pi_1(S^n) = 0$ for $n \geq 2$.

Proof:

Write S^n as $S^n = A_1 \cup A_2$ where A_1, A_2 are open and each homeomorphic to $\mathbb{R}^n$ (recall stereographic projection) and $A_1 \cap A_2$ is homeomorphic to $S^{n-1} \times \mathbb{R}$



Choose a basepoint $x_0 \in A_1 \cap A_2$

Let $n \geq 2$. Then $A_1 \cap A_2$ is path connected. Then by lemma 1.15 (Hatcher), every loop in S^n based at x_0 is homotopic to a product of loops in A_1 or A_2.

Since $\pi_1(A_1) = 0 = \pi_1(A_2)$ (as $A_1 \cong \mathbb{R}^n \cong A_2$), this product is nullhomotopic.

Theorem: <u>Fundamental Theorem of Algebra</u>

Every non-constant polynomial with coefficients in $\mathbb{C}$ has a root in $\mathbb{C}$.

Proof:

Consider an arbitrary polynomial $p(z) = z^n + a_1 z^{n-1} + a_2 z^{n-2} + \dots + a_n$.

Suppose, $p(z)$ has no roots in $\mathbb{C}$ (for contradiction)

Since $p(z)$ has no roots in $\mathbb{C}$, then $\forall r \in \mathbb{R}$,

$$f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$$ is a loop in $S^1 \subset \mathbb{C}$ based at 1.

$$\hookrightarrow f_r(0) = \frac{p(r)/p(r)}{|p(r)/p(r)|} = 1$$

$$f_r(1) = \frac{p(r\cos(2\pi) + r i \sin(2\pi))/p(r)}{(\dots)} = 1$$

Then, as r varies, f_r is a homotopy of loops with basepoint 1.

For $r = 0$, f_0 is the trivial loop constant at 1.

$\therefore [f_r] = 0 \quad \forall r$ in $\pi_1(S^1)$

$\therefore p(z) \longleftrightarrow 0 \in \pi_1(S^1) \longrightarrow$ 1

Now, consider a large r s.t $r > |a_1| + \dots + |a_n|$ and $r > 1$

Then, for $|z| = r$, $p(z)$ has no solution on $|z| = r$:

$$|z^n| > (|a_1| + \dots + |a_n|)|z^{n-1}| > |a_1 z^{n-1}| + \dots + |a_n| \geq |a_1 z^{n-1} + \dots + a_n|$$

$$\Rightarrow |z|^n > |a_1 z^{n-1} + \dots + a_n|$$

$\Rightarrow p_t(z) := z^n + t(a_1 z^{n-1} + \dots + a_n)$. $\forall t \in I$ has no root on the circle $|z| = r$ $\longrightarrow$ this is a deformation of our polynomial to z^n

Then, redefine $$f_r(s) := \frac{p_t(re^{2\pi i s})/p_t(r)}{|p_t(re^{2\pi i s})/p_t(r)|}$$

Let t go from 1 to 0, we find a homotopy from the loop f_r to $w_n(s) = e^{2\pi i n s}$.

But $[w_n] = [\omega]^n$ $\therefore$ $p(z) \longleftrightarrow n \in \pi_1(S^1) \longrightarrow$ (2)

$\Rightarrow [w_n] = [f_r] = 0$ using (1) and (2)

$\therefore n = 0$. $\longrightarrow$ contradiction as we assumed the degree was n.

Theorem: <u>Brouwer Fixed Point Theorem in 2 dimensions</u>

Every continuous map $h: D^2 \mapsto D^2$ has a fixed point i.e $x \in D^2$ s.t $h(x) = x$.

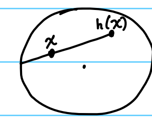
<u>Proof</u>:

Suppose, $\forall x \in D^2$, $h(x) \neq x$.

Define $r: D^2 \longmapsto S^1$ (where $\partial D^2 = S^1$)

to be the point where the line from $h(x)$ through x meets S^1:

$$r(x) = \frac{x - h(x)}{\| x - h(x) \|}$$



Clearly, r is continuous. Also, $r(x) = x$ $\forall x \in S^1$.

Thus, r is a retraction of D^2 onto S^1.

However, no such retraction exists.

$\hookrightarrow$ Let $f_0 \in \pi_1(S^1)$

In D^2, $f_0 \simeq$ constant loop by linear homotopy

$f_t(x) = (1-t) f_0(x) + t x_0$ $\leftarrow$ basepoint of f_0

Since $r = \text{id}$ on S^1, $r \circ f_t$ is a homotopy in S^1 from $r \circ f_0 = f_0$ to the constant loop at x_0, since r is a retraction of D^2 onto S^1.

But this contradicts the fact that $\pi_1(S^1)$ is non-zero.

Theorem: Borsak-Ulam Theorem in 2 dimensions

For every continuous map $f: S^2 \mapsto \mathbb{R}^2$, $\exists$ a pair of antipodal points x and $-x$ in S^2 s.t $f(x) = f(-x)$.

OR

Weather Theorem

At any moment, there exists a pair of antipodal points on Earth s.t they have the exact same temperature and pressure.

Proof:

Suppose not for $f: S^2 \longmapsto \mathbb{R}^2$

Define $g: S^2 \longmapsto S^1$ by $g(x) = \dfrac{f(x) - f(-x)}{\|f(x) - f(-x)\|}$. Notice: $-g(x) = g(-x)$.

Let the loop η in $S^2 \subseteq \mathbb{R}^3$ be $\eta(s) = (\cos(2\pi s), \sin(2\pi s), 0)$ → circles the equator of S^2 once.

and let $h: I \mapsto S^1$ be the composed loop $h = g \circ \eta$

Now, $g(-x) = -g(x) \implies h(s + \frac{1}{2}) = -h(s) \quad \forall s \in [0, \frac{1}{2}]$.

Now, the loop h can be lifted to $\tilde{h}: I \to \mathbb{R}$.

Since $h(s+\frac{1}{2}) = -h(s)$, $\tilde{h}(s + \frac{1}{2}) = \tilde{h}(s) + \frac{q}{2}$ for some odd integer.

Now, q is independent of s: q depends on $s \in [0, \frac{1}{2}]$ continuously but can take on odd integer values $\implies$ it must be constant.

Also, $\tilde{h}(1) = \tilde{h}(\frac{1}{2}) + \frac{q}{2} = \tilde{h}(0) + q$

$\therefore$ h represents q times the generator of $\pi_1(S^1)$

Since q is odd, h is not nullhomotopic

But $h = g \circ \eta : I \longrightarrow S^2 \longmapsto S^1$ and η is nullhomotopic in S^2

Shrink it along the surface.

$\therefore$ $g \circ \eta$ is null homotopic

$\therefore$ h is nullhomotopic $\longrightarrow$ contradiction.

Van Kampen Theorem

Free Product of Groups

First, we fix some notation:

(1) $G = \langle X | R \rangle$ is a group.

$X \longrightarrow$ set of generators

$R \longmapsto$ set of relations

Example 1: $G = \langle a, b \mid a^5 b^{-1} a b^3 = 1,\ b^7 a^9 = 1 \rangle$

$= \langle a, b \rangle \big/$ normal subgroup generated by $a^5 b^{-1} a b^3$, $b^7 a^9$

Example 2: $\mathbb{Z} = \langle g \rangle$

$\mathbb{Z}/n = \langle g \mid g^n \rangle$

(2) Product of groups:

Given a collection of groups G_α, $\alpha \in A$, the product is $\prod_{\alpha \in A} G_\alpha$ which can be regarded as functions $\alpha \longmapsto g_\alpha \in G_\alpha$.

↳ Suppose $(g_1, g_2, g_3, \ldots\ldots) \in \prod_{\alpha \in A} G_\alpha$

Then, this corresponds to a function f s.t $f(\alpha) = g_\alpha \in G_\alpha$. So, $f(1) = g_1$, $f(2) = g_2$, ...

↳ $(g_1, g_2, \ldots\ldots) \cdot (h_1, h_2, \ldots\ldots) = f \cdot h$

as $f(1) \cdot h(1) = g_1 \cdot h_1$, $f(i) \cdot h(i) = g_i \cdot h_i$

$\boxed{\#}$ Problem with direct sum $\bigoplus_\alpha G_\alpha$ or $\prod_\alpha G_\alpha$:

Elements of different subgroups G_α commute with each other.

Eg: $G_1 = \mathbb{Z}_2$

$G_2 = \mathbb{Z}_3$

$G_1 \times G_2 = \{(a, b) : a \in G_1, b \in G_2\}$

then, consider subgroups $\{0\} \times \mathbb{Z}_3$ and $\mathbb{Z}_2 \times \{0\}$

Let $x_1 = (1, 0) \in \mathbb{Z}_2 \times \{0\}$

$x_2 = (0, 1) \in \{0\} \times \mathbb{Z}_3$

$x_1 \cdot x_2 = (1, 0) \cdot (0, 1) = (1, 1) = (0, 1) \cdot (1, 0) = x_2 \cdot x_1$

As such, we will work with free products.

(3) <u>Free Product</u>:

$*_\alpha G_\alpha$ consists of elements of the form $g_1 g_2 \cdots g_m$ for finite $m \geq 0$ s.t:

(1) each $g_i \in G_{\alpha_i}$

(2) $g_i \neq 1_{G_i}$

(3) g_i and g_{i+1} belong to different groups (i.e $\alpha_i \neq \alpha_{i+1}$)

→ words "$g_1 g_2 \cdots g_m$" satisfying these conditions are called <u>reduced</u>

→ unreduced words can be simplified to reduced ones by writing adjacent letters in the same G_{α_i} as a single letter and by cancelling trivial letters.

→ empty word = identity of $*_\alpha G_\alpha$.

→ Group operation: $(g_1 \cdots g_m)(h_1 \cdots h_n) = g_1 \cdots g_m h_1 \cdots h_n$ and this should be simplified to reduced form ie if $g_m h_1 \in G_\alpha$, then write $(g_m h_1)$ as one letter and if it is identity, we cancel it.

Ex: $(g_1 \cdots g_m)(g_m^{-1} \cdots g_1^{-1})$ = identity / empty word.

<u>Associative</u>:

Let W be the set of reduced words $g_1 \cdots g_m$ including empty word.

For each $g \in G_\alpha$, we associate the function $L_g : W \longrightarrow W$ by multiplication on the left: $L_g(g_1 \cdots g_m) = g g_1 \cdots g_m$ (& simplify)

Property of this association $g \longmapsto L_g$ is that $\boxed{L_{gg'} = L_g L_{g'}}$ $\boxed{\text{for } g, g' \in G_\alpha}$ i.e $g(g'(g_1 \cdots g_m)) = (gg')(g_1 \cdots g_m)$ → this associativity follows from associativity in G_α.

Now $L_{gg'} = L_g L_{g'} \Rightarrow L_g$ is invertible with the inverse $L_{g^{-1}}$.

The association $g \longmapsto L_g$ is, thus, a homomorphism from G_α to the group $P(W)$ of all permutations of W. More generally, we can define:

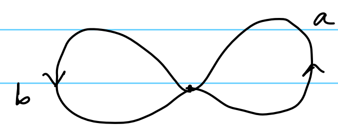
$L : W \longrightarrow P(W)$ by $L(g_1 \cdots g_m) = L_{g_1} \cdots L_{g_m}$ for each reduced word $g_1 \cdots g_m \in W$.

L is injective as the permutation $L(g_1 \cdots g_m)$ sends the empty word to $g_1 \cdots g_m$.

Now, the product operation in W corresponds under L to composition in $P(W)$ as $L_{gg'} = L_g L_{g'}$. Since composition in $P(W)$ is associative, the product in W is associative.

Eg:

(1) $\mathbb{Z} * \mathbb{Z}$:



Consider circles A and B at the basepoint x_0.

Suppose $\pi_1(A)$ is generated by a

$\pi_1(B)$ is generated by b

Then $\underbrace{a^5 b^2 a^{-3} b}$ is a loop in the $A \vee B$ described above

goes around A 5 times, around B 2 times, inverse around A 3 times. around B once

This is a word in $\mathbb{Z} * \mathbb{Z}$

Multiplication: $(b^4 a^5 b^2)(a^3 b^{-1} a) = b^4 a^5 b^2 a^3 b^{-1} a$

This is an example of a free group → the free product of any no. of copies of $\mathbb{Z}$ (can be infinite)

→ one generator for each $\mathbb{Z}$

→ the generators are called a basis for the free group

→ no. of basis elements = rank of the free group.

(2) $\mathbb{Z}_2 * \mathbb{Z}_2$ ⟶ not a free group

Here, $a^2 = b^2 = $ identity

$\mathbb{Z}_2 * \mathbb{Z}_2$ ⟶ alternating words like $a, b, ab, ba, aba, bab, \dots$ and empty word.

Consider $\varphi : \mathbb{Z}_2 * \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2$ which outputs the length of the word mod 2. Then φ is surjective, and its kernel is the set of words of even length. ⟶ These words of even length form an infinite cyclic subgroup generated by ab or $(ba) = (ab)^{-1} \in \mathbb{Z}_2 * \mathbb{Z}_2$.

called the infinite dihedral group.

(*) Now, for a free product $*_\alpha G_\alpha$, each group G_α can be identified with a subgroup of $*_\alpha G_\alpha$ consisting of the empty word and the non-identity one letter words $g \in G_\alpha$.

$\rightarrow$ $\therefore$ the empty word is the common identity element of all the subgroup G_α (which are otherwise disjoint).

(*) A consequence of associativity is that any product $g_1 \cdots g_m$ of elements $g_i \in G_\alpha$ has a unique reduced form.

Proposition :

for the free product $*_\alpha G_\alpha$, any collection of homomorphisms $\varphi_\alpha : G_\alpha \longmapsto H$ extends uniquely to a homomorphism

$$\varphi : *_\alpha G_\alpha \longmapsto H$$

s.t $\varphi(g_1 \cdots g_n) = \varphi_{\alpha_1}(g_1) \cdots \varphi_{\alpha_n}(g_n)$

Example : for a free product $G * H$, the inclusions $G \hookrightarrow G \times H$ and $H \hookrightarrow G \times H$ induce a surjective homomorphism $G * H \longrightarrow G \times H$.

Amalgamated Free Product

$G_1, G_2, H \longrightarrow$ groups

$$\left.\begin{aligned} f_1 &: H \longrightarrow G \\ f_2 &: H \longrightarrow G \end{aligned}\right\} \text{homomorphism}$$

Amalgamated free product : $G_1 *_H G_2 = G_1 * G_2 \,/\, f_1(h) = f_2(h) \ \forall h \in H$

Ex:

$$\underset{\langle g_1 \rangle}{\mathbb{Z}} *_{\mathbb{Z}} \underset{\langle g_2 \rangle}{\mathbb{Z}} = \langle g_1, g_2 \mid g_1^n = g_2^m \rangle$$

$$f_1(h) = g_1^n$$

$$f_2(h) = g_2^m$$

Van Kampen's Theorem

Suppose, the space X can be decomposed as the union of a collection of path-connected, open subsets A_α, each of which contains the basepoint $x_0 \in X$.

Consider the inclusion $A_\alpha \hookrightarrow X$ which induces the homomorphisms

$$j_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X).$$

This can be extended to the homomorphism

$$\Phi : \ast_\alpha \pi_1(A_\alpha) \longmapsto \pi_1(X)$$

$$\text{s.t. } \Phi(f_1 f_2 \cdots f_n) = j_{\alpha_1}(f_1) \cdots\cdots j_{\alpha_n}(f_n)$$

Consider the inclusion $A_\alpha \cap A_\beta \hookrightarrow A_\alpha$ inducing $i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \longmapsto \pi_1(A_\alpha)$.

Then, $j_\alpha i_{\alpha\beta}(\omega) = j_\beta i_{\beta\alpha}(\omega)$ for any loop in $A_\alpha \cap A_\beta$.

and both of them are induced by the inclusion

$$A_\alpha \cap A_\beta \hookrightarrow X.$$

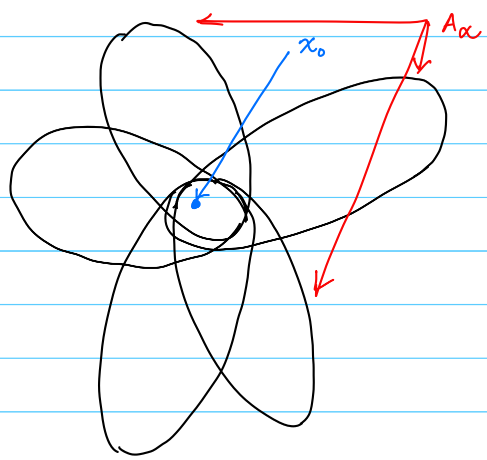
$\therefore$ kernel of Φ contains all elements of the form

$i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$.

$\hookrightarrow \Phi(i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1})$

$= \Phi(\text{empty word})$) since we are going to define it to be 1

$= \text{constant loop}$ (since Φ



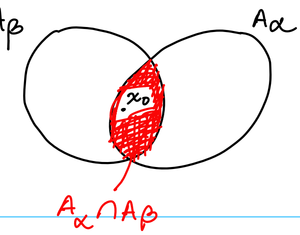
$$A_\alpha \hookrightarrow X$$
$$\Downarrow$$
$$j_\alpha : \pi_1(A_\alpha) \longrightarrow \pi_1(X)$$

$$A_\alpha \cap A_\beta \hookrightarrow A_\alpha$$
$$\Downarrow$$
$$i_{\alpha\beta} : \pi_1(A_\alpha \cap A_\beta) \longmapsto \pi_1(A_\alpha)$$

$$\Phi : \ast_\alpha \pi_1(A_\alpha) \longmapsto \pi_1(X)$$

by

$$\Phi(f_1 f_2 \cdots f_n) = j_{\alpha_1}(f_1) \cdots\cdots j_{\alpha_n}(f_n)$$

(1)  (2)

Theorem: Seifert-van Kampen Theorem

(1) If X is the union of path connected open sets A_α each containing the basepoint $x_0 \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then the homomorphism

$$\Phi : \ast_\alpha \pi_1(A_\alpha) \longrightarrow \pi_1(X)$$

{a loop in X can be thought of as composition of loops in each A_α

is surjective.

(2) In addition, if each intersection $A_\alpha \cap A_\beta \cap A_\gamma$ is path connected, then the kernel of Φ is the normal subgroup N generated by elements of the form $i_{\alpha\beta}(\omega)\, i_{\beta\alpha}(\omega)^{-1}$ for $\omega \in \pi_1(A_\alpha \cap A_\beta)$ and, hence, Φ induces an isomorphism

$$\pi_1(X) \approx \ast_\alpha \pi_1(A_\alpha)/N.$$

Proof:

(1) is true by the following:

Lemma:
If a space X is the union of a collection of path connected open sets A_α, each containing the basepoint $x_\alpha \in X$ and if each intersection $A_\alpha \cap A_\beta$ is path connected, then every loop in X at x_0 is homotopic to a product of loops each of which is contained in a single A_α.

(2). We need to prove that $\ker(\Phi)$ is N.

Def: Factorization of a loop

factorization of $[f] \in \pi_1(X)$ is a formal product $[f_1] \cdots [f_k]$

s.t (1) each $f_i \in A_\alpha$ for some α at basepoint x_α and $[f_i] \in \pi_1(A_\alpha)$

(2) the loop f is homotopic to $f_1 \cdots f_k$ in X.

The factorization of $[f]$ is a word in $\ast_\alpha \pi_1(A_\alpha)$, possibly unreduced that is mapped to $[f]$ by Φ.

Surjectivity of Φ is equivalent to saying that every $[f] \in \pi_1(X)$ has a factorization.

Def: Equivalent Factorizations

Two factorizations are equivalent if they are related by sequences of the following two moves or their inverses:

(move 1): combine adjacent terms $[f_i][f_{i+1}]$ into $[f_i \cdot f_{i+1}]$ if $[f_i]$, $[f_{i+1}]$ both belong to the same $\pi_1(A_\alpha)$

(move 2): regard $[f_i] \in \pi_1(A_\alpha)$ as lying in $\pi_1(A_\beta)$ instead if f_i is a loop in $A_\alpha \cap A_\beta$

move 1 does not change the element in $\ast_\alpha A_\alpha$ w.r.t the definition of factorization

move 2 does not change the image of this element in the quotient group $\boxed{Q := \ast_\alpha \pi_1(A_\alpha)/N}$

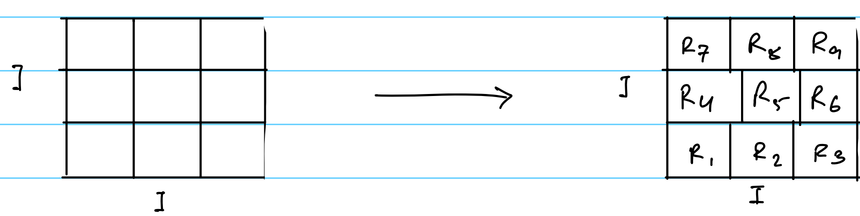
We want to prove that any two factorizations of f are equivalent. Then, we will have proven that $Q \mapsto \pi_1(X)$ is injective $\Rightarrow$ N is the kernel of Φ $\Rightarrow Q \approx \pi_1(X)$.

Let $[f_1] \cdots [f_k]$ and $[f_1'] \cdots [f_\ell']$ be two factorizations of $[f]$. Then, the composed paths $f_1 \cdots f_k$ and $f_1' \cdots f_\ell'$ are homotopic via $F: I \times I \to X$.

Now, $\exists$ partitions $0 = s_0 < s_1 < \cdots < s_m = 1$ and $0 = t_1 < t_2 < \cdots < t_n = 1$ s.t each rectangle $R_{ij} = [s_{i-1}, s_i] \times [t_{j-1}, t_j]$ is mapped by F into a single A_α called A_{ij} $\longrightarrow$ we get these partitions by covering $I \times I$ by finitely many rectangles $[a,b] \times [c,d]$ each mapping to a single A_α and then partitioning $I \times I$ by the union of all vertical and horizontal lines containing edges of these rectangles.

$\llcorner$ The s-partition subdivides these partitions to give the products $f_1 \cdots f_k$ and $f_1' \cdots f_\ell'$.

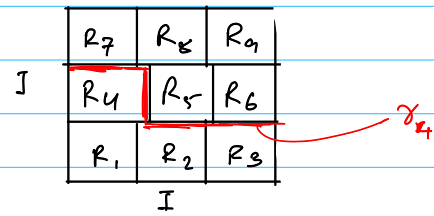
Now, F maps a nbhd of R_{ij} to A_{ij}, so we may perturb the vertical sides of the rectangles R_{ij} so that each point in $I \times I$ is in at most three R_{ij}'s:



We are perturbing only the middle rows (not the first and last — we are assuming there are at least three).
Label the rectangles $R_1, R_2, \ldots, R_{mn}$.

If γ is a path in $I \times I$ from the left to the right edge, then the restriction $F|_\gamma$ is a loop at the basepoint x_0 since F maps both the left and right edges of $I \times I$ to x_0.
Let γ_r be the path separating the first r rectangles

from the rest.



Then, γ_0 is the bottom edge of $I \times I$

γ_{mn} is the top edge.

We go from γ_r to γ_{r+1} by pushing across the rectangle R_{r+1}.

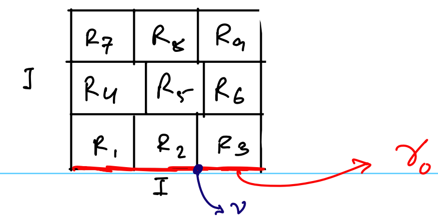
Now, consider the vertices of R_r. For each vertex v with $F(v) \neq x_0$, we choose a path g_v from x_0 to $F(v)$. that lies in the intersection of the two or three A_{ij}'s corresponding to the R_r's containing v. (for a visualization, see the proof the surjectivity).

Then, we have a factorization of $[F|_{\gamma_r}]$ by inserting the appropriate paths $\bar{g}_v g_v$ into $F|_{\gamma_r}$. at successive vertices (similar to the way we did it in the proof of surjectivity).

This factorization depends on our choices: consider the path between two successive vertices which can lie in 2 different A_{ij}'s since the path may be in 2 different R_{ij}'s. However, different choices of A_{ij}'s here gives equivalent factorizations (using move 2).

Also, the factorization for successive paths γ_r and γ_{r+1} are equivalent since pushing γ_r across R_{r+1} to γ_{r+1} changes $F|_{\gamma_r}$ to $F|_{\gamma_{r+1}}$ by a homotopy within the A_{ij} corresponding to R_{r+1} and we can choose this A_{ij} for all the segments of γ_r and γ_{r+1} in R_{r+1}.

This shows that the factorization associated with all γ_r are equivalent.



We can arrange so that the factorization associated to γ_0 is equivalent to the factorization $[f_1]\cdots[f_k]$ by choosing the path g_v for each vertex v along the lower edge of $I\times I$ to lie not just in the two A_{ij}'s corresponding to the R_s's containing v but also in the A_α for the f_i containing v in its domain.

↳ in case v is the common endpoint of the domains of two ~~com~~ f_i's, $F(v)=x_0$, so there is no need to choose a g_v here.

Similarly, assume that the factorization associated to the final γ_{mn} is equivalent to $[f_1']\cdots[f_\ell']$.
Since the factorization associated to all the γ_r's are equivalent, $[f_1]\cdots[f_k]$ and $[f_1']\cdots[f_\ell']$ are equivalent.

Seifart Van-Kampen; in amalgamated free product notation:

$$\pi_1(X) := \pi_1(U) \underset{\pi_1(U \cap V)}{*} \pi_1(V)$$

provided U, V are open + path connected, $X = U \cup V$, $U \cap V$ = path connected

More generally,

subject to this equivalence relation

$$\boxed{\begin{array}{c} \pi_1(X) = \underset{\alpha}{*}\, \pi_1(U_\alpha) \Big/ \Big((i_{\alpha\beta})_*(\omega) = (i_{\beta\alpha})_*(\omega),\ \forall \omega \in \pi_1(U_\alpha \cap U_\beta) \Big)_{\forall \alpha, \beta} \\ i_{\alpha\beta} : U_\alpha \cap U_\beta \hookrightarrow U_\alpha \\ i_{\beta\alpha} : U_\alpha \cap U_\beta \hookrightarrow U_\beta \end{array}}$$

$\hookrightarrow$ Loops in $U_\alpha \cap U_\beta$ must be interpreted the same way, regardless of whether we see them as loops in U_α or U_β.

Example

(i) $\mathbb{Z} \underset{\mathbb{Z}}{*} \mathbb{Z} = \mathbb{Z}$

as ~~we~~ our equivalence relation is that the generators of the two copies of $\mathbb{Z}$ are equivalent, so we really have only one copy of $\mathbb{Z}$

Applying Van Kampen's Theorem to compute fundamental groups:

(1) Wedge sum of X_α: $\pi_1\left(\bigvee_\alpha X_\alpha\right)$

Let the basepoints be $x_\alpha \in X_\alpha$.

For each $x_\alpha \in X_\alpha$, if x_α is a deformation retract of an open neighbourhood $U_\alpha \subset X_\alpha$, then X_α is a deformation retract of its open neighborhood $A_\alpha = X_\alpha \vee \bigvee_{\beta \neq \alpha} U_\beta$

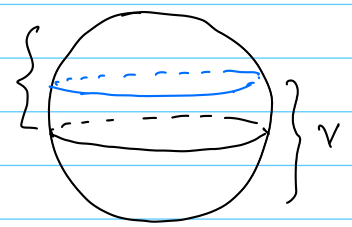
$\therefore X_\alpha \simeq A_\alpha$. Here, A_α will be our open cover.

The intersection of two or more distinct X_α is $\bigvee_\alpha U_\alpha$, which deformation retracts to a point where we wedge all X_α. So, the intersection of A_α is trivial as it is trivially path connected. This also means N is the trivial subgroup.

$$\therefore \quad \pi_1\left(\bigvee_\alpha X_\alpha\right) \cong \mathop{*}_\alpha \pi_1(X_\alpha)$$

$$\rightarrow \quad \pi_1\left(\bigvee_\alpha S^1_\alpha\right) \cong \mathop{*}_\alpha \pi_1(S^1_\alpha) = \mathop{*}_\alpha \mathbb{Z} \longrightarrow \text{the free group}$$

(2) $\pi_1(S^n)$:



Note $U \cong B^n$, $V \cong B^n$, $U \cap V = S^{n-1} \times I \simeq S^{n-1}$

For $n \geq 2$:

$$\pi_1(S^n) = \pi_1(U) \mathop{*}_{\pi_1(S^{n-1})} \pi_1(V)$$

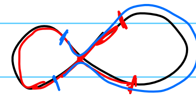
$$= \pi_1(B^n) \mathop{*}_{\pi_1(S^{n-1})} \pi_1(B^n)$$

$$= 1 \mathop{*}_{\pi_1(S^{n-1})} 1$$

$$= 1$$

(for $n < 2$, S^{n-1} is not path connected: $S^0 = \{-1, 1\}$)

(3) $\underbrace{\pi_1(\infty)}_{S^1 \vee S^1}$



$\therefore \pi_1(\infty) = \pi_1(\propto) \underset{\pi_1(\times)}{*} \pi_1(\infty) = \mathbb{Z} \underset{1}{*} \mathbb{Z} = F_2$

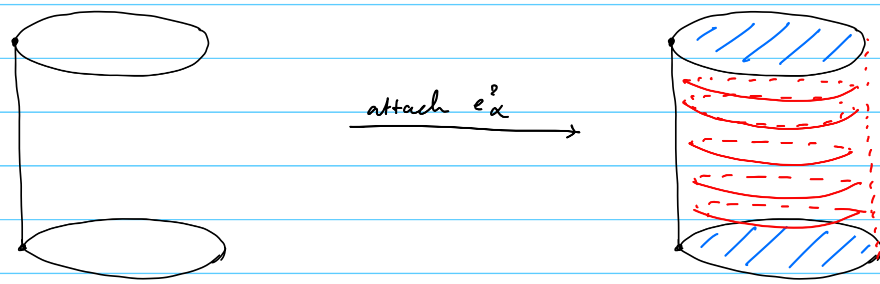
$\propto \simeq \bigcirc$

Applying Van-Kampen's Theorem to Cell Complexes.

Intuition:

Consider a path connected space X.
Suppose, we attach a bunch of 2-cells e^2_α to X via $\varphi_\alpha : S^1 \mapsto X$
(since the boundary of e^2_α is S^1). Let the basepoint of S^1 be s_0.
Then, $\varphi_\alpha : S^1 \mapsto X$ is a loop at $\varphi_\alpha(s_0)$.
Call this loop φ_α $\longrightarrow$ although a loop would be $f: I \to X$ we use the shorthand $\varphi_\alpha : S^1 \mapsto X$ to refer to this loop at $\varphi_\alpha(s_0)$.

For each α, we get a different loop at each $\varphi_\alpha(s_0)$ since the basepoints $\{\varphi_\alpha(s_0) : \forall \alpha\}$ may not all be the same.
To fix this, we choose a basepoint $x_0 \in X$ and a path γ_α in X from $x_0 \in X$ to $\varphi_\alpha(s_0)$ for each α. Then, $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$ is a loop at $\varphi_\alpha(s_0)$, for each α.
These loops may not be nullhomotopic in X but they will be after the cell e^2_α is attached.



!!! $\therefore$ The normal subgroup $N \subset \pi_1(X, x_0)$ generated by all the loops $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$ for each α lies in the kernel of the map $\pi_1(X, x_0) \longrightarrow \pi_1(Y, x_0)$ induced by the inclusion $i : X \hookrightarrow Y$

Proposition :

(a) If Y is obtained from X by attaching 2 cells as described, then the inclusion $i: X \hookrightarrow Y$ induces a surjection $\pi_1(X, x_0) \twoheadrightarrow \pi_1(Y, x_0)$. whose kernel is N.

$\therefore \quad \pi_1(Y) \approx \pi_1(X)/N$

(b) If Y is obtained from X by attaching n-cells for a fixed $n > 2$, then the inclusion $i: X \hookrightarrow Y$ induces an isomorphism

$$\pi_1(Y) \approx \pi_1(X).$$

(c) for a path connected cell complex X, the inclusion of the 2-skeleton $i: X^2 \hookrightarrow X$ induces an isomorphism

$$\pi_1(X^2, x_0) \longrightarrow \pi_1(Y, x_0).$$

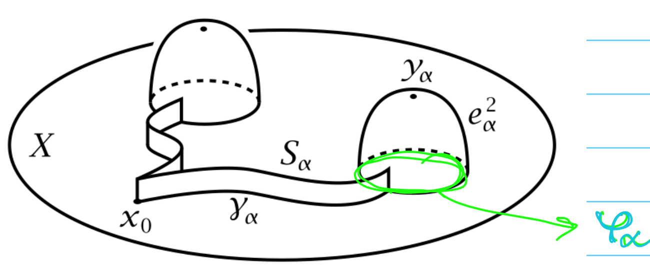
Note : in (a), N is independent of the choice of our paths γ_α since if $\gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha}$ is in N, and $\eta_\alpha \varphi_\alpha \overline{\eta_\alpha}$ is another possible path, then $(\eta_\alpha \overline{\gamma_\alpha}) \gamma_\alpha \varphi_\alpha \overline{\gamma_\alpha} \overline{(\eta_\alpha \overline{\gamma_\alpha})} = \eta_\alpha \varphi_\alpha \overline{\eta_\alpha}$ i.e they are conjugate to each other.

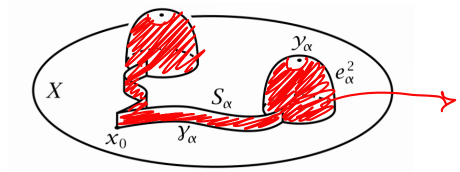
Proof :

(a) Suppose, Y is obtained from X by attaching 2 cells.
Expand Y to a slightly larger space Z s.t Z def retracts onto Y. (so, $\pi_1(Z) \approx \pi_1(Y)$)

↳ build Z by doing : attach rectangular strips $S_\alpha = I \times I$ with the lower edge $I \times \{0\}$ attached along γ_α and the right edge $\{1\} \times I$ attached along an arc starting from $\varphi_\alpha(s_0)$ and going radially into e^2_α and the left edges of every strip (ie for each α) identified together.

→ Since the top edge is not attached to anything, we can def retract Z onto Y.





$\to A \cap B$

Suppose, in each 2-cell e^2_α, we choose a basepoint y_α s.t. y_α is not in the arc along which S_α is attached.

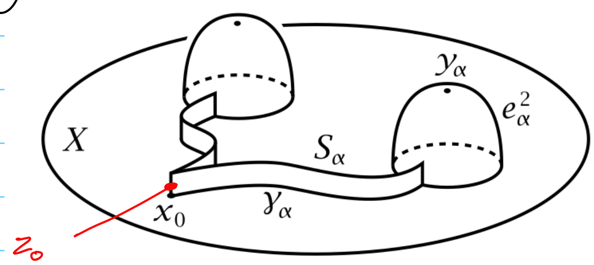
Let $A = Z - \bigcup_\alpha \{y_\alpha\}$ $\longrightarrow$ this def retracts onto X $\therefore A \simeq X$

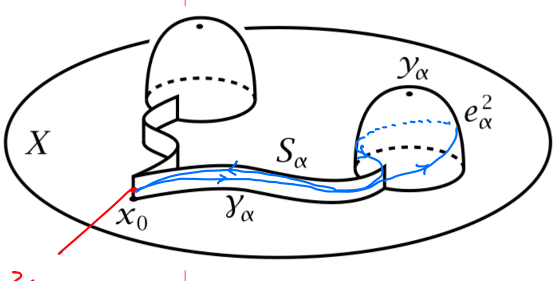
$B = Z - X$ $\longrightarrow$ contractible $\Rightarrow \pi_1(B) = 0$

$\therefore \pi_1(Y) \approx \pi_1(Z) \approx \pi_1(A)/N \approx \pi_1(X)/N \longrightarrow$ a normal subgroup generated by loops in $A \cap B$

Now, cover Z by $A \cup B$. Since $\pi_1(B) = 0$, $\therefore \pi_1(Z) \approx \pi_1(A)/N$ where N is the generated by the image of the map $\pi_1(A \cap B) \to \pi_1(A)$ since B is contractible

$\hookrightarrow$ specifically, let $z_0 \in A \cap B$ near x_0 on the the segment where all S_α intersect





Now, choose loops $\delta_\alpha \in \pi_1(A \cap B, z_0)$ based at z_0 representing elements in $\pi_1(A, z_0)$ that correspond to $[\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha] \in \pi_1(A, x_0)$ after a basepoint shift from x_0 to z_0 along the edge connecting all S_α.

$\hookrightarrow$ $\therefore$ we make these loops $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$ generate N:

recall: the normal subgroup generated by is $(i_{\alpha\beta})_*(\omega)(i_{\beta\alpha})_*(\omega)$ for $\forall \omega \in \pi_1(A \cap B)$ B is contractible
Since we are making each δ trivial in the quotient

Claim: $\pi_1(A \cap B, z_0)$ is generated by loops δ_α.

$\longrightarrow$ Use Van Kampen's Theorem again but to the cover of $A \cap B$ by open sets $\boxed{A_\alpha = A \cap B - \bigcup_{\beta \neq \alpha} e^2_\beta}$

Given A_α deformation retracts onto a circle in $e^2_\alpha - \{y_\alpha\}$,

$\pi_1(A_\alpha, z_0) \approx \mathbb{Z}$ generated by δ: $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$

but ~~thus~~

we already saw prior to the theorem:

> $\therefore$ The normal subgroup $N \subset \pi_1(X, x_0)$ generated by all the loops $\gamma_\alpha \varphi_\alpha \bar{\gamma}_\alpha$ for each α lies in the kernel of the map $\pi_1(X, x_0) \to \pi_1(Y, x_0)$ induced by the inclusion $i: X \hookrightarrow Y$

(b) Same proof as before, but here A_α def retracts onto a sphere S^{n-1} so, $\pi_1(A_\alpha) = 0$ if $n > 2$ (as $\pi_1(S^n) = 0$ for $n \geq 2$)

$\therefore \pi_1(A \cap B) = 0 \Rightarrow$ the normal subgroup generated by loops in $\pi_1(A \cap B)$ is trivial.

(c) follows from (b) by induction when X is finite dimensional, ie $X = X^n$.

(recall ~~X~~ we go from X^2 to X by attach e^n_α for $n > 2$)

Now, suppose X is not finite dimensional.

Let $f: I \to X$ be a loop at the basepoint $x_0 \in X^2$. The image of f is compact which must lie in X^n for some n.

Then, by part (b), f is homotopic to a loop in X^2.

↳ as $X^2 \hookrightarrow X^n$ is surjective $\Rightarrow f$ is homotopic to a loop in X^2

Thus, $\pi_1(X^2, x_0) \longrightarrow \pi_1(X, x_0)$ is surjective

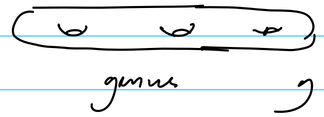
Claim: $\pi_1(X^2, x_0) \longrightarrow \pi_1(X, x_0)$ is ~~surjective~~ also injective

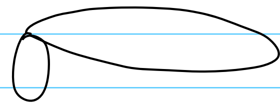
Suppose f is a loop in $\pi_1(X^2, x_0)$ that is nullhomotopic in X via $F: I \times I \longrightarrow X$.

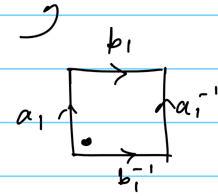
Then F has a compact image lying in some X^n and we can assume $n > 2$.

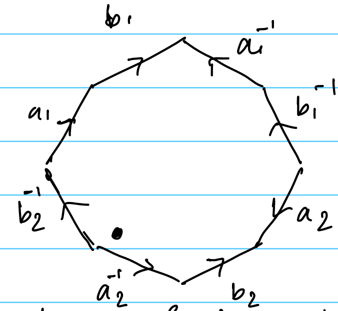
Since $\pi_1(X^2, x_0) \longrightarrow \pi_1(X^n, x_0)$ is injective by (b), f is nullhomotopic in X^2.

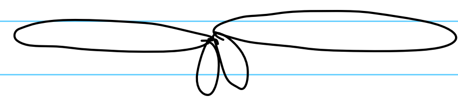
(4) Orientable surface of genus g:

Σ_g : 

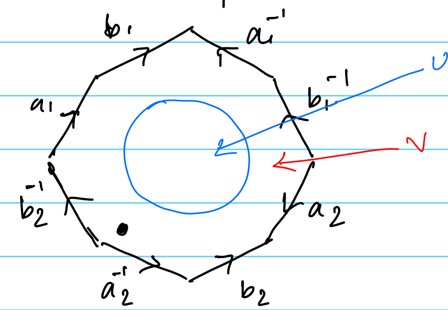
Σ_1 : 





Σ_2 : 

Generally, Σ_g can be constructed from a polygon of $4g$ sides labelled $a_1, b_1, a_1^{-1}, b_1^{-1}, \ldots\ldots, a_g, b_g, a_g^{-1}, b_g^{-1}$ and gluing a_i to a_i^{-1}, b_i to b_i^{-1}. Glue them so that all loops meet at the same basepoint



$U \cap V = S^1 \Rightarrow \pi_1(S^1) \cong \mathbb{Z}$

$V = B^2 \Rightarrow \pi_1(B^2) \cong 1$ as B^2 is contractible

$$\pi_1(U) = \pi_1\Big(\bigvee_{2g} S^1\Big)$$

$$\therefore \pi_1(\Sigma_g) = \pi_1\Big(\bigvee_{2g} S^1\Big) *_{\mathbb{Z}} 1$$

$$\pi_1(\Sigma_g) = \langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g \mid \underbrace{[a_1, b_1]}_{a_1 b_1 a_1^{-1} b_1^{-1}} [a_2 b_2] \cdots [a_g, b_g] = 1 \rangle$$

Another way to see this is as from Hatcher:

As a first application we compute the fundamental group of the orientable surface M_g of genus g. This has a cell structure with one 0-cell, $2g$ 1-cells, and one 2-cell, as we saw in Chapter 0. The 1-skeleton is a wedge sum of $2g$ circles, with fundamental group free on $2g$ generators. The 2-cell is attached along the loop given by the product of the commutators of these generators, say $[a_1, b_1] \cdots [a_g, b_g]$. Therefore

$$\pi_1(M_g) \approx \langle\, a_1, b_1, \cdots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \,\rangle$$

where $\langle g_\alpha \mid r_\beta \rangle$ denotes the group with generators g_α and relators r_β, in other words, the free group on the generators g_α modulo the normal subgroup generated by the words r_β in these generators.

Corollary:

The surface M_g is not homotopy equivalent or homeomorphic to M_h if $g \neq h$.

(5) Non-orientable surface of genus g:

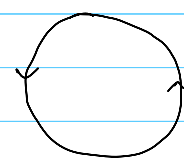
we create them by $\longrightarrow$ (1) take the wedge sum of g circles

$$\pi_1\left(\bigvee_g S^1\right) = \underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{g \text{ times}}$$

let the generator of ~~ea~~ each of these groups be a_i. So, the generators are $a_1, a_2, \ldots\ldots, a_g$.

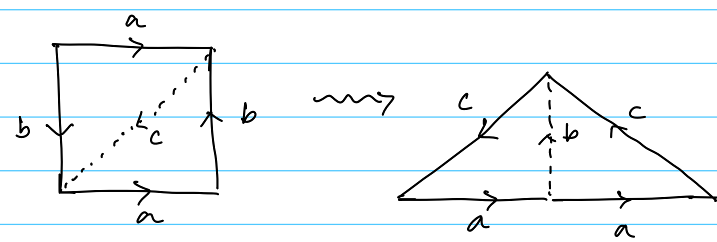
(2) attach a 2-cell to this wedge sum along the path $a_1^2 a_2^2 \ldots\ldots a_g^2$

N_1: $\mathbb{R}P^2$ $\longrightarrow$



generated by a_1

N_2: Klein bottle $\longrightarrow$

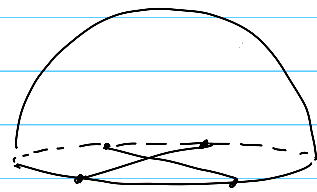


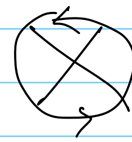
cut the square along c & then reassemble

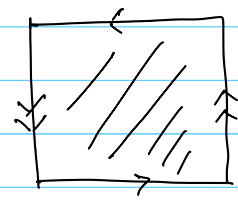
By our proposition,

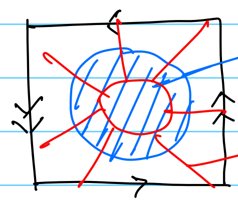
$$\pi_1(N_g) \approx \langle a_1, \ldots\ldots, a_g \mid a_1^2 \ldots\ldots a_g^2 \rangle$$

(5) $\mathbb{RP}^2 \longrightarrow$

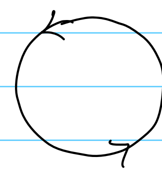


$\mathbb{RP}^2 = D^2 \cup$ 

$=$ 



$U = D^2 \longrightarrow$ trivial

$V =$  $\simeq S^1/\sim$

$U \cap V = S^1$

Then, $\pi_1(\mathbb{RP}^2) =$

(b) Non-orientable surface of genus g:

$$\pi_1(N_g) = \pi_1\left(\right. $$  $$\left. \right) = \langle a_1, \dots\dots, a_g \mid a_1^2 a_2^2 \cdots a_g^2 = 1 \rangle$$

Then, $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$

Corollary:

(a) $\Sigma_g \not\cong \Sigma_h$ for $g \neq h$

(b) $N_g \not\cong N_h$ for $g \neq h$

(c) $\Sigma_g \not\cong N_g$

Proof: We want to say that they have different fundamental groups.
First, we do Abelianisation: Given G a group,

$$\mathrm{Ab}\, G := G/[G,G]$$

$\hookrightarrow$ generated by all $[g,h]$

Classification for finitely generated abelian groups:

$$\mathbb{Z} \oplus \dots\dots \oplus \mathbb{Z} \oplus \mathbb{Z}/p_1^{m_1} \oplus \mathbb{Z}/p_2^{m_2} \oplus \dots \oplus \mathbb{Z}/p_k^{m_k}$$

$p_i \to$ prime, not necessarily distinct.

Now, (a) $\mathrm{Ab}\, \pi_1(\Sigma_g) = \langle a_1 b_1 \dots a_g b_g \rangle / 1 = \mathbb{Z}^{2g}$

Notice $\mathbb{Z}^{2g} \neq \mathbb{Z}^{2h}$ for $g \neq h$

(b) $\mathrm{Ab}\, \pi_1(N_g) : \langle a_1 \dots a_g \mid 2a_1 + 2a_2 + \dots + 2a_g = 0 \rangle$

$a_i \to$ commute

Let $b = a_1 + \dots + a_g$

$$\mathrm{Ab}\, \pi_1(N_g) = \langle a_1, \dots\dots, a_{g-1}, b \mid 2b = 0 \rangle$$
$$= \mathbb{Z}^{g-1} \oplus \mathbb{Z}/2$$

More generally.

Let X be path connected,

Attach an n-cell $\rightsquigarrow Y$

Then, $\pi_1(Y) = \pi_1(X) *_{\pi_1(S^{n-1})} 1 = \pi_1(X)$ if $n>2$

$= $ quotient of $\pi_1(X)$ if $n=2$

$$\pi_1(\mathbb{RP}^n) = \pi_1\big(\underbrace{e^0 \cup e^1 \cup e^2}_{\mathbb{RP}^2} \cup e^3 \cup \cdots \cup e^n\big)$$

Suppose $n \geq 2$

$= \pi_1(\mathbb{RP}^2)$

$= \mathbb{Z}/2$

$$\pi_1(\mathbb{CP}^n) = \pi_1\big(\underbrace{e^0 \cup e^2}_{S^2} \cup \cdots \cup e^{2n}\big)$$

$= \pi_1(S^2)$

$= 1$

Corollary

For every group G. $\exists$ a 2 dimensional cell complex X_G with $\pi_1(X_g) \approx G$.

Proof :

Let $G = \langle g_\alpha | r_\beta \rangle$ be a representation. This exists as every group is a quotient of a free group

↳ start with a free group F generated by the set $\{g_\alpha\}$ without any relations other than group axioms. So, F is just words that can not be reduced.

Thus, $G = F/N$ where N = normal closure of $\{r_\beta\}$

So, $r_\beta = \ker(\Phi)$ where $\Phi : F \mapsto G$.

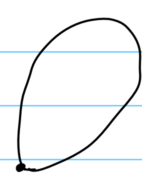
Construct X_G from $\bigvee_\alpha S^1_\alpha$ by attaching 2-cells e^2_β by the loops specified by the word r_β.

→ each circle corresponds to a generator g_α

Example :

(1) $\mathbb{Z}_2 = \langle g \,|\, g^2 = 1 \rangle$

↳ step 1 :



← a loop corresponding to the generator g.

step 2 : attach a disk so that as we trace the boundary of the disk, we trace this loop twice.

$\therefore$ antipodal points are glued.

$\therefore$ we get $\mathbb{RP}^2$ and $\mathbb{Z}_2 \approx \mathbb{RP}^2$

Covering Spaces

Goal: Classify covering spaces of X in terms of $\pi_1(X)$.
Algebraic aspects of the fundamental group $\longleftrightarrow$ geometric language of covering spaces

Review:

Def: Covering Spaces

$p: \tilde{X} \longmapsto X$ is covering means $\forall x \in X$, $\exists$ nbhd U of x s.t U is evenly covered i.e $p^{-1}(U) = \coprod_\alpha V_\alpha$
s.t $p|_{V_\alpha}: V_\alpha \longmapsto U$ is a homeomorphism.
$U \longrightarrow$ called "evenly covered", $V_\alpha \longrightarrow$ called sheets of $\tilde{X}$ over U.

$\rightarrow$ If U is connected, V_α are the connected components of $p^{-1}(U)$.

$\rightarrow$ when U is not connected, the decomposition of $p^{-1}(U)$ may not be unique.

$\rightarrow$ p need not be surjective as $p^{-1}(U)$ can be empty.

Examples:
(1) $p: \mathbb{R} \longrightarrow S^1$ by $p(s) = e^{2\pi i s} \in S^1$

Some results we have already proven...
Provided covering spaces $p: \tilde{X} \longmapsto X$

Called the homotopy lifting property

(1) Given a map $F: Y \times I \longmapsto X$ and a map $\tilde{F}: Y \times \{0\} \longmapsto \tilde{X}$ that lifts $F|_{Y \times \{0\}}$ (i.e.. $p \circ \tilde{F} = F|_{Y \times \{0\}}$ on $Y \times \{0\}$), there exists a unique map $\tilde{F}: Y \times I \longmapsto \tilde{X}$ lifting $F: Y \times I \longmapsto X$ (i.e. $p \circ \tilde{F} = F$ on $Y \times I$) s.t it agrees on $Y \times \{0\}$

called the path lifting property

(2) for each path $f: I \longmapsto X$ s.t $f(0) = x_0 \in X$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lift $\tilde{f}: I \longmapsto \tilde{X}$ s.t $\tilde{f}(0) = \tilde{x}_0$ and so, $f = p \circ \tilde{f}$

path homotopy lifting property

(3) for each homotopy $f_t: I \longmapsto X$ of paths s.t $f_t(0) = x_0$ and each $\tilde{x}_0 \in p^{-1}(x_0)$, $\exists$ a unique lifted homotopy $\tilde{f}_t: I \longrightarrow \tilde{X}$ of paths s.t $\tilde{f}_t(0) = \tilde{x}_0$.
$\therefore$ $f_t = p \circ \tilde{f}_t$

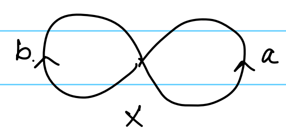
More examples of covering spaces

(1) Consider $S \subset \mathbb{R}^3$ consisting of points $(s\cos(2\pi t), s\sin(2\pi t), t)$ for $(s,t) \in (0,\infty) \times \mathbb{R}$

Then $p : S \longmapsto \mathbb{R}^2 - \{0\}$ via the map $(x,y,z) \longmapsto (x,y)$

(2) $p : S^1 \longmapsto S^1$ via $p(z) = z^n$, $z \in \mathbb{C}$ with $|z| = 1$
$n \in \mathbb{Z}_{>0}$.

(3) Covering spaces of $S^1 \vee S^1 =: X$



Consider X to be a graph with one vertex and edges a and b $\rightarrow$ a and b have some orientation.

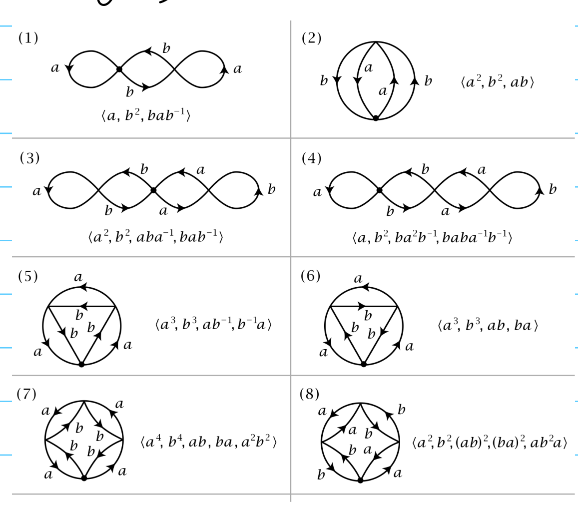
Now, consider $\tilde{X}$ to be another graph with four ends of edges at each vertex, similar to X, and, again, each edge is labelled either a or b and oriented in a way that there is

1 a-edge going away
1 a-edge going into
1 b-edge going away
1 b-edge going into

can be >1 vertices

Call $\tilde{X}$ a 2-oriented graph.

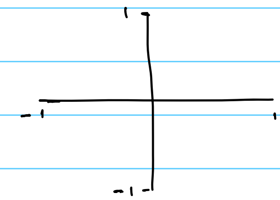
Examples :



Then, we can construct $p: \tilde{X} \to X$ s.t p sends all vertices of $\tilde{X}$ to the one vertex in X and sending each edge of $\tilde{X}$ to an edge in X with the same label s.t p is a homeomorphism from the interior of the edge and preserves orientation

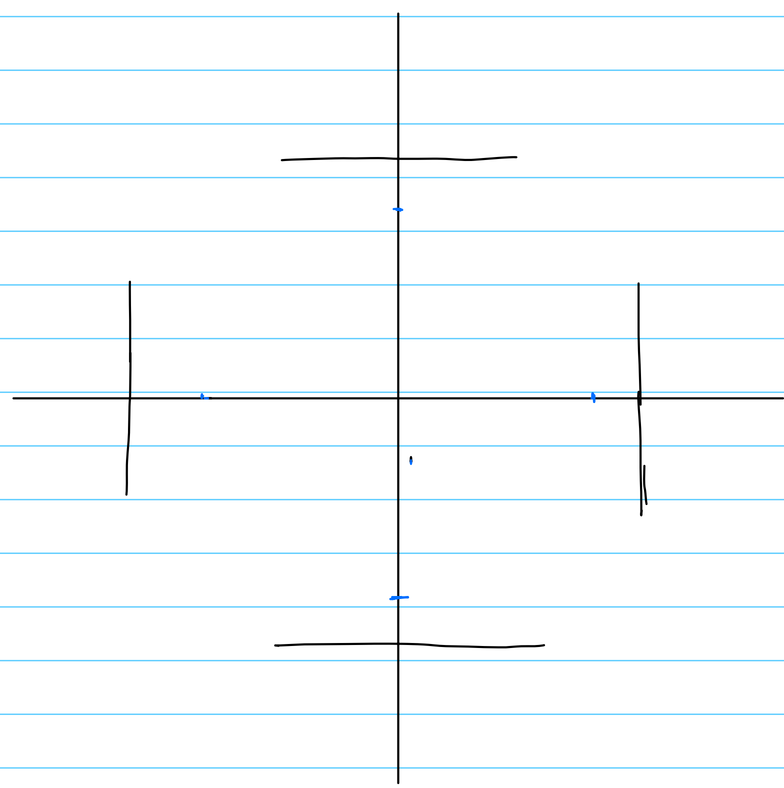
(4) $X = S^1 \vee S^1$

We start with $(-1,1)$ in the axes of $\mathbb{R}^2$



Next, select a fixed $\lambda = \frac{1}{3}$.

Adjoin 4 open segments of length 2λ at distance λ from the ends of the previous segments and perpendicular to them



Then add perpendicular open segments of length $2\lambda^2$ at distance λ^2 from the endpoints of previous segments

At the n^{th} stage, add ~~op~~ perpendicular open segments of length $2\lambda^{n-1}$ at distance λ^{n-1} from the endpoints.

Lifting Properties

#1 **Proposition** : Homotopy Lifting Property (HLP) / Covering Homotopy Property
Given a covering space $p: \tilde{X} \mapsto X$, a homotopy $f_t: Y \mapsto X$ and a map $\tilde{f}_0: Y \mapsto \tilde{X}$ lifting f_0, then there exists a unique homotopy $\tilde{f}_t: Y \mapsto \tilde{X}$ of $\tilde{f}_0$ that lifts f_t.

Proof :

We already proved:

Given a map $F: Y \times I \mapsto X$ and a map $\tilde{F}: Y \times \{0\} \mapsto \tilde{X}$ that lifts $F|_{Y\times\{0\}}$ (i.e. $p \circ \tilde{F} = F|_{Y\times\{0\}}$ on $Y \times \{0\}$), there exists a unique map $\tilde{F}: Y \times I \mapsto \tilde{X}$ lifting $F: Y \times I \mapsto X$ (i.e. $p \circ \tilde{F} = F$ on $Y \times I$) s.t. it agrees on $Y \times \{0\}$

#1 By the path lifting property's uniqueness, every lift of the constant path is constant.

Proposition 1.31: (a) The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \mapsto \pi_1(X, x_0)$ induced by $p: (\tilde{X}, \tilde{x}_0) \mapsto (X, x_0)$ is injective

(b) $Im(p_*(\pi_1(\tilde{X}, \tilde{x}_0)))$ in $\pi_1(X, x_0)$ consists of a homotopy class of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

$$Im(p_*) = \{ [f] \in \pi_1(X, x_0) \mid f \text{ lifts to a loop at } \tilde{x}_0 \}$$

Proof :

(a) $\ker(p_*)$ consists of loops s.t each belongs to the homotopy class of a loop $\tilde{f}_0 : I \mapsto \tilde{X}$ with a homotopy $f_t : I \mapsto X$ of $f_0 = p\,\tilde{f}_0$ to the trivial loop f_1.
Then, we can find a homotopy $\tilde{f}_t : I \mapsto \tilde{X}$ ~~s.t f~~ starting from $\tilde{f}_0$ and ending with the constant loop (since the lift of the constant loop is constant) $\rightarrow$ we do this using path homotopy lifting property

$\therefore [\tilde{f}_0] = 0$ in $\pi_1(\tilde{X}, \tilde{x}_0) \Rightarrow p_*$ is injective

(b) $\supseteq$: this is obvious as if f lifts to a loop $\tilde{f}$ in $\tilde{x}_0$
then, $p_*(\tilde{f}) = p \circ \tilde{f} = f$ by definition

$\subseteq$: Suppose $[f] \in \pi_1(X, x_0)$ ~~is~~ represents an element of the image of p_*.

Then $[f] \simeq$ a loop in the image of p_*.

Let $[f] \simeq \underbrace{p_*([g])}_{\text{a loop that lifts to a loop in } \tilde{X}}$

Then by path homotopy lifting, $[f]$ itself ~~to~~ can be lifted to a loop in $\tilde{X}$.

Proposition :

The no. of sheets of a covering space $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ with X and $\tilde{X}$ path-connected equals the index of $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ in $\pi_1(X, x_0)$

index $\hookrightarrow$ no. of distinct cosets of the subgroup

Proof :

Let g be a loop based at x_0 in X

Let $\tilde{g}$ be the lift of g in $\tilde{X}$ starting at $\tilde{x}_0$.

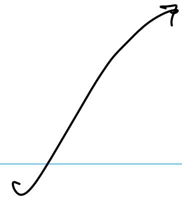
A product $h \cdot g$ with $[h] \in H = p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ has the lift $\tilde{h} \cdot \tilde{g}$ ending at the same point as $\tilde{g}$ since $\tilde{h}$ is a loop.

$\therefore$ We can define Φ from cosets $H[g]$ (with $[g] \in \pi_1(X, x_0)$) to $p^{-1}(x_0)$ by sending $H[g]$ to $\tilde{g}(1)$.

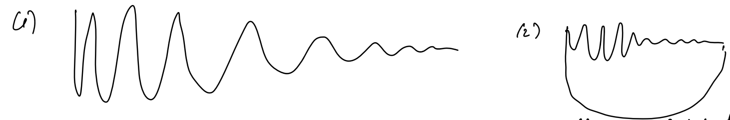
→ Φ is surjective : $\tilde{X}$ is path connected, so $\tilde{x}_0$ can be joined to any point in $p^{-1}(x_0)$ by a path $\tilde{g}$ projecting to a loop g at x_0

→ Φ is injective : $\Phi(H[g_1]) = \Phi(H[g_2])$

$\Rightarrow$ $g_1 \cdot \bar{g}_2$ lifts to a loop in $\tilde{X}$ based at $\tilde{x}_0$, so $[g_1][g_2]^{-1} \in H \Rightarrow \therefore H[g_1] = H[g_2]$



Non-example



path connected but not l.p.c

Exercise

(i) locally path connected + connected $\Rightarrow$ path connected.

Def: Locally path connected

A space Y is called locally path connected if $\forall y \in Y$, $\forall$ nbhd U of y, $\exists V \subset U$ open, $y \in V$, V is path connected.

Proposition: (Lifting Criterion)

Suppose we have a covering space $p : (\tilde{X}, \tilde{x}_0) \longrightarrow (X, x_0)$ and a map $f : (Y, y_0) \longrightarrow (X, x_0)$ with Y path connected and locally path-connected. Then, a lift $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$.

Proof:

If $\exists$ a lift $\tilde{f} : (Y, y_0) \longrightarrow (\tilde{X}, \tilde{x}_0)$, of f, then

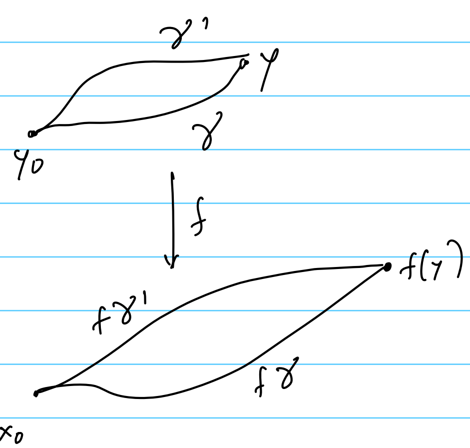
$$p \circ \tilde{f} = f$$
$$f_* = p_* \cdot \tilde{f}_*$$

So, $f_* \in \operatorname{Im} p_*$

Conversely, let $y \in Y$ and let γ be a path in Y from y_0 to y. The path $f\gamma$ in X from x_0 has the unique lift (by path lifting property) $\widetilde{f\gamma}$ starting from $\tilde{x}_0$.

Let $\tilde{f}(y) = \widetilde{f\gamma}(1)$

This is $\longrightarrow$ well-defined:



$f(y_0) = x_0$

Let γ' be a $\underline{\text{different}}$ path from y_0 to y. Then $(f\gamma') \cdot \overline{(f\gamma)}$ is a loop at x_0 as shown in the diagram on the left. Call the loop $h_0 := (f\gamma') \cdot \overline{(f\gamma)} : I \longrightarrow X$

Here $\underbrace{[h_0] \in f_*(\pi_1(Y, y_0))}_{\text{by the def as } h_0 \text{ is defined by a loop in } X} \subset \underbrace{p_*(\pi_1(\tilde{X}, \tilde{x}_0))}_{\text{by hypothesis}}$

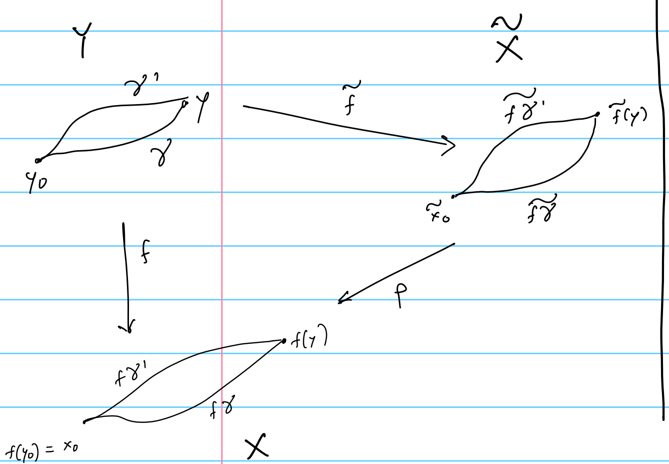
$\therefore$ There exists a homotopy h_t from h_0 to a loop h_1 in X. By path lifting property, h_1 lifts to $\tilde{h}_1$ in $\tilde{X}$ based at $\tilde{x}_0$. Then, by path homotopy lifting, h_t lifts to $\tilde{h}_t$.

Now, $\tilde{h}_1$ is a loop based at $\tilde{x}_0$, so $\tilde{h}_0$ is also a loop based at $\tilde{x}_0$ $\longrightarrow$ in fact $\tilde{h}_t$ are all loops at $\tilde{x}_0$.

By uniqueness of lifted paths, the first half of $\tilde{h}_0$ is $\widetilde{f\gamma'}$ and the second half is $\overline{(\widetilde{f\gamma})}$ and the common midpoint is ~~$f\gamma$~~

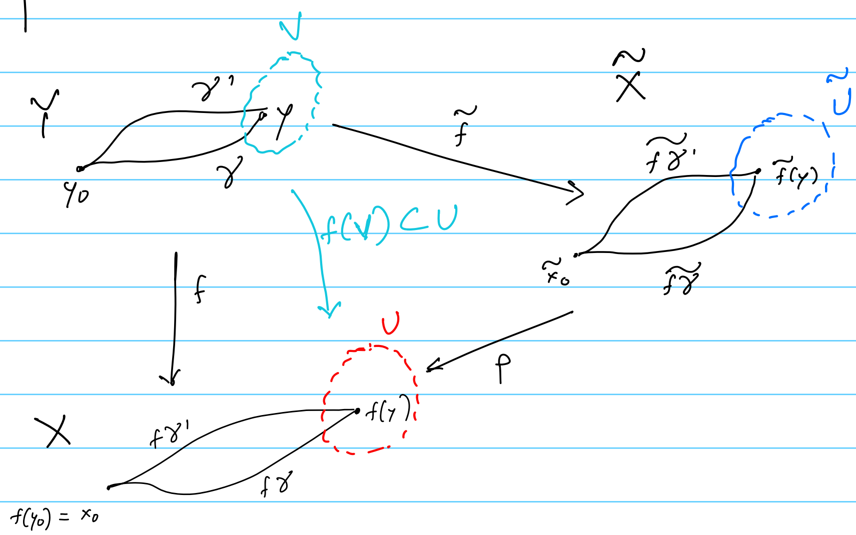
$$\widetilde{f\gamma'}(1) = \widetilde{f\gamma}(1)$$

$\therefore$ $\tilde{f}(y)$ is well-defined



$\longrightarrow$ This map is continuous:

Let $U \subset X$ be an open neighborhood of $f(y)$ having a lift $\tilde{U} \subset \tilde{X}$ containing $\tilde{f}(y)$ s.t $p: \tilde{U} \to U$ is a homeomorphism. Now, choose a path connected open nbhd V of y with $f(V) \subset U$.



For paths from y_0 to points $y' \in V$, take a path γ from y_0 to y (as above) and then a path η from y to $y' \in V$ inside V.

Then, the paths $(f\gamma)\cdot(f\eta)$ in X have lifts $(\widetilde{f\gamma})\cdot(\widetilde{f\eta})$ where

$$\widetilde{f\eta} = p^{-1} f\eta \quad \text{and} \quad p^{-1}: U \to \tilde{U}$$

$$\therefore \tilde{f}(V) \subset \tilde{U} \quad \text{and} \quad \tilde{f}|_V = p^{-1} f$$

$\therefore$ $\tilde{f}$ is continuous at y

$$\begin{tikzcd} Y \arrow[r, bend left, "\tilde{f}_1"] \arrow[r, "\tilde{f}_2"'] \arrow[d, "f"'] & \tilde{X} \arrow[dl, "p"] \\ X & \end{tikzcd}$$

<u>Proposition</u>: <u>Unique Lifting Criterion</u>

Given a covering space $p: \tilde{X} \mapsto X$ and a map $f: Y \mapsto X$, if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \mapsto \tilde{X}$ of f agree at one point of Y and Y is connected, then $\tilde{f}_1$ and $\tilde{f}_2$ agree on all of Y.

Proof: For a point $y \in Y$, let U be an openly covered nbhd of $f(y)$ in X. Then $p^{-1}(U) = \coprod_{\alpha} \tilde{U}_\alpha$ s.t $U_\alpha \cong U$

sheets

Let $\tilde{U}_1$ and $\tilde{U}_2$ be the sheets containing $\tilde{f}_1(y)$ and $\tilde{f}_2(y)$ respectively.

By continuity of $\tilde{f}_1$ and $\tilde{f}_2$, $\exists$ a neighbourhood $N \ni y$ s.t $\tilde{f}_1$ maps N to $\tilde{U}_1$

and $\tilde{f}_2$ maps N to $\tilde{U}_2$

Now, if $\tilde{f}_1(y) \neq \tilde{f}_2(y)$, then $\tilde{U}_1 \neq \tilde{U}_2 \Rightarrow \tilde{U}_1$ and $\tilde{U}_2$ are disjoint

& $\tilde{f}_1 \neq \tilde{f}_2$ throughout N

Take the union of all such N for every point $y \in Y$ where $\tilde{f}_1 \neq \tilde{f}_2$, then the complement (where they agree) is closed.

If $\tilde{f}_1(y) = \tilde{f}_2(y)$, then $\underbrace{\tilde{U}_1 = \tilde{U}_2}_{\text{as each } U_\alpha \text{ is disjoint}} \Rightarrow \tilde{f}_1 = \tilde{f}_2$ on N

because; $p\tilde{f}_1 = p\tilde{f}_2$

as both are equal to f

and p is injective on $\tilde{U}_1 = \tilde{U}_2$

Here N is open. Take the union of all such N, we get an open set.

$\therefore$ The set of points where $\tilde{f}_1 = \tilde{f}_2$ is both open and closed in Y

Since Y is connected, this set is all of Y.

Universal Cover $\longrightarrow$ a simply connected covering space of X. where X is path-connected + locally path connected.

Constructing a covering space that is simply connected

Assumptions:

(1) X is path-connected. By "components", we refer to components.

(2) X is locally path connected

$\therefore$ Covering spaces $\tilde{X}$ are also locally path connected.

(3) X is semilocally simply connected (defined below)

Galois Correspondence

Given a covering space $p: (\tilde{X}, \tilde{x}_0) \longmapsto (X, x_0)$,

we can have a corresponding subgroup

$$p_*(\pi_1(\tilde{X}, \tilde{x}_0)) \subset \pi_1(X, x_0).$$

$\rightarrow$ is this function surjective? i.e is every subgroup in $\pi_1(X, x_0)$ realised as $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ for some p?

In particular, is the trivial subgroup of $\pi_1(X, x_0)$ realised?

Since p_* is injective, this is the same as asking $\rightarrow$ does X have a simply connected $\tilde{X}$

$\hookrightarrow$ because then $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ will be trivial subgroup in $\pi_1(X, x_0)$ as p_* is injective

Def: Semilocally Simply-connected

Each point $x \in X$ has a neighbourhood U s.t the inclusion-induced map $i_*: \pi_1(U, x) \longmapsto \pi_1(X, x)$ is trivial. (ie $i_*(\pi_1(U, x)) = 1$)

Lemma: (Necessary condition for $\tilde{X}$ to be simply connected)

If X has a covering space $\tilde{X}$ that is simply connected then X is semilocally simply connected.

Proof: Suppose $p: \tilde{X} \longmapsto X$ where $\tilde{X}$ is simply connected.

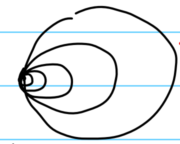
$\forall x \in X$, $\exists U \ni x$ with a lift $\tilde{U} \subset \tilde{X}$ s.t $\tilde{U} \underset{p}{\cong} U$.

Each loop in U lifts to a loop in $\tilde{U}$.

The lifted loop is nullhomotopic in $\tilde{X}$ as $\pi_1(\tilde{X}) = 0$ as $\tilde{X}$ is simply connected. $\therefore$ $p \circ$ (nullhomotopic loop) is a nullhomotopic loop in X

Non-example of semilocally simply connected: (1) Shrinking edge of circles

↳ wedge circles with radii $\frac{1}{n}$ for $n = 1, 2, \ldots$ centered at $\left(\frac{1}{n}, 0\right)$



Hawaiian ring!

(2) The cone $CX = (X \times I)/(X \times \{0\})$ of the shrinking wedge of circles is semilocally simply connected but not locally simply connected.

<u>Lemma</u>: If X is <u>locally simply connected</u>, then X is semilocally simply-connected.

<u>Recall</u>: CW complexes are locally contractible

$\therefore$ CW complexes are semilocally simply connected.

Theorem :

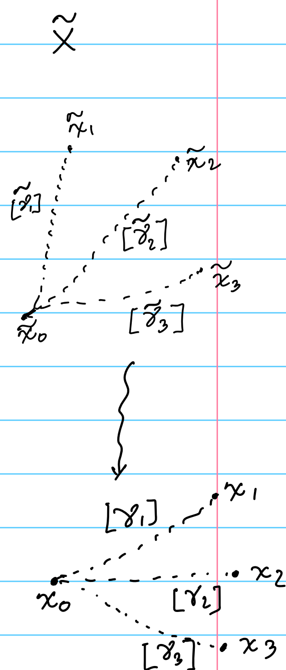
If X is path-connected, locally path connected and semilocally simply connected.

Then, X has a universal cover

Proof : we see how to construct a simply-connected covering space $\tilde{X}$ if X is → path-connected + locally path-connected + semilocally simply connected.

↳ path connected + fundamental group is trivial

⇒ ∀ points $\tilde{x} \in \tilde{X}$ can be connected to $\tilde{x}_0$ can be connected by a unique homotopy class of paths in $\tilde{X}$ starting at $\tilde{x}_0$ (by Hatcher prop. 1.6)



Motivation :

Since $\tilde{X}$ is simply-connected, each point $\tilde{x} \in \tilde{X}$ can be thought of as a homotopy class of paths starting at $\tilde{x}_0$. By path homotopy lifting property, homotopy classes of paths in $\tilde{X}$ from $\tilde{x}_0$ are the same as homotopy classes of paths in X starting at x_0.

∴ we can describe $\tilde{X}$ purely in terms of X.

Constructing $\tilde{X}$ s.t $\tilde{X}$ is simply-connected covering space of X

Assume X is path-connected + locally path connected + semilocally simply connected.

Let $x_0 \in X$ be a basepoint of X.

→ Define $\tilde{X} := \{ [\gamma] \mid \gamma \text{ is a path in } X \text{ starting at } x_0 \}$

$[\gamma]$ ↳ homotopy class of paths starting at $\gamma(0)$ and ending at $\gamma(1)$

→ Define $p : \tilde{X} \longrightarrow X$ s.t $p([\gamma]) = \gamma(1)$ is well-defined.

Given X is path connected, the endpoints $\gamma(1)$ can be any point of X

∴ $\boxed{p \text{ is surjective}}$

Recall : $\boxed{p_* \text{ is always injective}}$

Now, we need a suitable topology on X and $\tilde{X}$.

Properties:

(1) Define, $\mathcal{U} := \{ U \subset X \mid U$ is open + path-connected & $\pi_1(U) \to \pi_1(X)$ is trivial $\}$

mapped to the constant loop.

Let $\mathcal{U}$ be a collection of ~~subsets~~ path-connected open sets $U \subset X$ s.t $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial $\longrightarrow$ note: if $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial for some basepoint in U, then it is trivial for all choices of basepoints as U is path-connected.

Since X is s.s.c, every point is inside a $U \in \mathcal{U}$

$\longrightarrow$ note: a path connected subset $V \subset U \in \mathcal{U}$ is also in $\mathcal{U}$ since the composition $\pi_1(V) \longrightarrow \pi_1(U) \longrightarrow \pi_1(X)$ will also be trivial.

So, $\mathcal{U}$ is a basis for the topology on X if X is locally path-connected + semilocally simply connected.
$\hookrightarrow$ Easy to check using the conditions:

Theorem 3.3.9. *Let X be a set and $\mathcal{B}$ a collection of subsets of X satisfying the following two conditions:*

(1) $\forall x \in X, \quad \exists B \in \mathcal{B}$, s.t. $x \in B$;

(2) If $x \in B_1 \cap B_2, \quad B_1, B_2 \in \mathcal{B} \;\Rightarrow \exists B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subseteq B_1 \cap B_2$.

Then $\mathcal{B}$ is a base for a topology on X, namely

$$\mathcal{T}_{\mathcal{B}} = \{ U \subseteq X \mid \forall x \in U \quad \exists B \in \mathcal{B} \quad s.t. \quad x \in B \subseteq U \}.$$

So far, we have $\tilde{X}$, p and a basis on X.

Next, we construct a topology on $\tilde{X}$.

Now, consider a $U \in \mathcal{U}$. Then, $p^{-1}(U) = \{ [\gamma] \mid \gamma(1) \in U \}$. Pick $u_0 \in U$ and a path in X from x_0 to u_0.

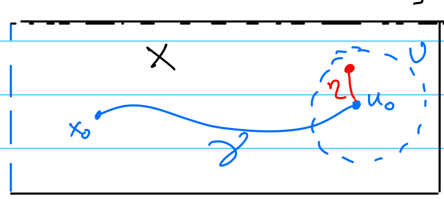
Let $U_{[\gamma]} = \{ [\gamma \cdot \eta] \mid \eta$ is a path in $U \}$

Then, $U_{[\gamma]} \cong U$

(2) Define, for $\forall U \in \mathcal{U}$ and a path γ in X starting from x_0 to a point u_0 in U, the set

$$U_{[\gamma]} := \{ [\gamma \cdot \eta] \mid \eta \text{ is a path in } U \text{ with } \eta(0) = \gamma(1) \}$$

Now, $p : U_{[\gamma]} \longmapsto U$ is surjective - as $U \subset X$ is path connected, is injective as different choices of η s.t η joins a fixed x to $\gamma(1)$ are all homotopic as $\pi_1(U) \longrightarrow \pi_1(X)$ is trivial.



(3) $U_{[\gamma]} = U_{[\gamma']}$ if $[\gamma'] \in U_{[\gamma]}$.

$\longrightarrow$ if $\gamma' = \gamma \cdot \eta$, then elements of $U_{[\gamma']}$ have the form $[\gamma \cdot \eta \cdot \mu]$ $\therefore$ they all lie in $U_{[\gamma]}$. On the other hand, elements in $U_{[\gamma]}$ are $[\gamma \cdot \mu] = [\gamma \cdot \eta \cdot \bar{\eta} \cdot \mu]$ which is $[\gamma' \cdot \bar{\eta} \cdot \mu] \in U_{[\gamma']}$.

(4) The sets $U_{[\gamma]}$ form a basis for a topology on $\tilde{X}$.

$\longrightarrow$ given 2 sets $U_{[\gamma]}$ and $V_{[\gamma']}$ and

$[\gamma''] \in U_{[\gamma]} \cap V_{[\gamma']}$, we have that

$U_{[\gamma]} = U_{[\gamma'']}$ and $V_{[\gamma']} = V_{[\gamma'']}$

So, if $W \in \mathcal{U}$ s.t. $W \subset U \cap V$

and $\gamma''(1) \in W$ $(\gamma''(0) = x_0)$,

then $W_{[\gamma'']} \subset U_{[\gamma'']} \cap V_{[\gamma'']}$

and $[\gamma''] \in W_{[\gamma'']}$

Now,

$p : \tilde{X} \longmapsto X$ is

(a) a homeomorphism as it is a bijection between $V_{[\gamma']} \subset U_{[\gamma]}$ and the sets $V \in \mathcal{U}, V \subset U$

$p(V_{[\gamma']}) \overset{\text{s.t.}}{=} V$ (as p is surjective)

$p^{-1}(V) \cap U_{[\gamma]} = V_{[\gamma']}$ for any $[\gamma'] \in U_{[\gamma]}$ with endpoint in V

because

$V_{[\gamma']} \subset U_{[\gamma']} = U_{[\gamma]}$

and $V_{[\gamma']}$ maps onto V by p

(b) $\therefore$ p is continuous.

(c) p is a covering space.

$\hookrightarrow$ for fixed $U \in \mathcal{U}$, the sets $U_{[\gamma]}$ for varying $[\gamma]$ partition $p^{-1}(U)$

because if $[\gamma''] \in U_{[\gamma]} \cap U_{[\gamma']}$

then $U_{[\gamma]} = U_{[\gamma'']} = U_{[\gamma']}$

(d) $\tilde{X}$ is simply connected.

path-connected → for a point $[\gamma] \in \tilde{X}$, let γ_t be the path that equals γ in X on $[0,t]$ and is constant at $\gamma(t)$ from $[t,1]$.

Then, $t \longmapsto [\gamma_t]$ is a path in $\tilde{X}$ lifting γ that starts at $[x_0]$ and ends at $[\gamma]$.

Given $[\gamma]$ was an arbitrary point in $\tilde{X}$, this shows $\tilde{X}$ is path connected

$\pi_1(\tilde{X}) = 1$ → we show p_* maps $\pi_1(\tilde{X}, \tilde{x}_0)$ to 0.

Suffices since p_* is injective $\Rightarrow \pi_1(\tilde{X}, \tilde{x}_0) = 0$

We know $t \longmapsto [\gamma_t]$ lifts γ starting at $[x_0]$

For this lifted path to be a loop, $[\gamma_1] = [x_0]$

Since $\gamma_1 = \gamma$ and $[x_0]$ is constant,

$[\gamma] = [x_0]$

$\therefore \gamma$ is nullhomotopic

$U_{[\gamma]} := \{ [\gamma \cdot \eta] \mid \eta$ is a path in U with $\eta(0) = \gamma(1) \}$

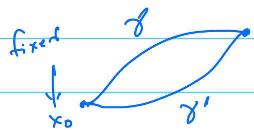
Here $X_H = \{ [\gamma] \mid \gamma(0) = x_0 \}$

Proposition:

Suppose X is path connected, locally path connected and semilocally simply connected.
Then, for any subgroup $H \subset \pi_1(X, x_0)$, $\exists$ a covering space $p_H : X_H \to X$ s.t. $p_*(\pi_1(X_H, \tilde{x}_0)) = H$ for a suitably chosen basepoint $\tilde{x}_0 \in X_H$.

$\tilde{X} := \{ [\gamma] \mid \gamma$ is a path in X starting at $x_0 \}$

Proof: First, construct $\tilde{X}$ as above
For points $[\gamma]$ and $[\gamma']$ in the simply-connected covering space $\tilde{X}$, define $[\gamma] \sim [\gamma']$ to mean $\gamma(1) = \gamma'(1)$
and $[\gamma \cdot \bar{\gamma}'] \in H$



$\llcorner$ Equivalence relation:
(a) reflexive as H contains the identity
(b) symmetric as H is closed under inverses
(c) transitive as H is closed under multip.

Now, let X_H be the quotient space of $\tilde{X}$ obtained by identify $[\gamma]$ with $[\gamma']$ if $[\gamma] \sim [\gamma']$

$X_H = \tilde{X} / [\gamma] \sim [\gamma']$ for $\forall [\gamma], [\gamma'] \in \tilde{X}$

$\llcorner$ note, if $\gamma(1) = \gamma'(1)$, then
$[\gamma] \sim [\gamma']$ iff $[\gamma \cdot \eta] \sim [\gamma' \cdot \eta]$

$X_H = \{ [\gamma] \mid \gamma(0) = x_0 \} / \left(\begin{matrix} [\gamma] \sim [\gamma'] \ \forall \\ [\gamma], [\gamma'] \in \tilde{X} \end{matrix} \right)$

$\Downarrow$
$\therefore$ Any two points in the basic neighborhoods $U_{[\gamma]}$ and $U_{[\gamma']}$ are identified in X_H
$\Rightarrow U_{[\gamma]} = U_{[\gamma']}$ in X_H

(Note, $H = 1 \rightsquigarrow X_H = \tilde{X}$
$H = \pi_1(X) \rightsquigarrow X_H = X$)

$\therefore$ the projection $p : X_H \to X$ by $p([\gamma]) = \gamma(1)$ is a covering space

Choose a basepoint $\tilde{x}_0 \in X_H$ to be the equivalence class of the constant path c at x_0. The image of
$p_* : \pi_1(X_H, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ is H

$\operatorname{Im}(p_*) = H$

$Im(p*) \subseteq H$: If $\gamma \in Im(p*)$, then γ lifts to a loop at $\tilde{x}_0 = [c]$ and $[\gamma] \in \pi_1(X, x_0)$

Now the lift of γ is starting at $[c]$ and ends at $[\gamma]$ (as $p([\gamma]) = \gamma(1)$)

Then $[\gamma] \sim [c]$ in $X_H \Rightarrow \gamma \in H$

$H \subseteq Im(p*)$: let $\gamma \in H \subset \pi_1(X, x_0)$

We can lift it to a loop in X_H using the def of p and X_H. Let that loop in X_H be $f(t)$

$$p^*(f) = p \circ f(t) = \gamma$$

so, $\gamma \in Im(p_*)$

Recall:

Proposition: 1.31 (a) The map $p_* : \pi_1(\tilde{X}, \tilde{x}_0) \longrightarrow \pi_1(X, x_0)$ induced by $p : (\tilde{X}, \tilde{x}_0) \mapsto (X, x_0)$, is injective

(b) $Im(p_*(\pi_1(\tilde{X}, \tilde{x}_0)))$ in $\pi_1(X, x_0)$ consists of a homotopy class of loops in X based at x_0 whose lifts to $\tilde{X}$ starting at $\tilde{x}_0$ are loops.

$$Im(p*) = \{ [f] \in \pi_1(X, x_0) \mid f \text{ lifts to a loop at } \tilde{x}_0 \}$$

?

Classification of Covering Spaces

Def: Isomorphism between covering spaces
$p_1 : \tilde{X}_1 \mapsto X$ and $p_2 : \tilde{X}_2 \mapsto X$ are isomorphic ~~with a~~ if $\exists$ a homeomorphic $f : \tilde{X}_1 \mapsto \tilde{X}_2$ s.t $p_1 = p_2 f$

$\rightarrow$ $\therefore f$ preserves the covering space structure taking $p_1^{-1}(x)$ to $p_2^{-1}(x)$ for each $x \in X$.
$\rightarrow$ The inverse f^{-1} is also an isomorphism
Composition of isomorphisms is an isomorphism $\Rightarrow$ this $\sim$ is an equivalence relation.

We can fix basepoints $\rightarrow$ i.e fix $\tilde{x}_1 \in \tilde{X}_1$, $\tilde{x}_2 \in \tilde{X}_2$, we say they are basepoint-preserving isomorphism

Proposition: (Isomorphic covering spaces) (Classification theorem part 1)
If X is path connected and locally path connected, then two path-connected covering spaces $p_1 : \tilde{X}_1 \mapsto X$ and $p_2 : \tilde{X}_2 \mapsto X$ are isomorphic via an isomorphism $f_1 : \tilde{X}_1 \mapsto \tilde{X}_2$ taking basepoint $\tilde{x}_1 \in p_1^{-1}(x_0)$ to a basepoint $\tilde{x}_2 \in p_2^{-1}(x_0)$

if and only if

$$p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$$

subgroups of $\pi_1(X, x_0)$

Proof:
If $\exists$ an isomorphism $f : (\tilde{X}_1, \tilde{x}_1) \rightarrow (\tilde{X}_2, \tilde{x}_2)$,
then $p_1 = p_2 f$ and $p_2 = p_1 f^{-1}$

$\therefore$ $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$
(as $(p_1)_* = (p_2)_* (f)_* \Rightarrow Im((p_1)_*) \subset Im((p_2)_*)$ & the same for the other direction)

Conversely suppose $p_{1*}(\pi_1(\tilde{X}_1, \tilde{x}_1)) = p_{2*}(\pi_1(\tilde{X}_2, \tilde{x}_2))$.
Then, by the lifting criterion;

Proposition: (Lifting Criterion)
Suppose we have a covering space $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ and a map $f : (Y, y_0) \rightarrow (X, x_0)$ with Y path connected and locally path-connected. Then, a lift $\tilde{f} : (Y, y_0) \rightarrow (\tilde{X}, \tilde{x}_0)$ of f exists if and only if $f_*(\pi_1(Y, y_0)) \subset p_*(\pi_1(\tilde{X}, \tilde{x}_0))$

We have $p_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (X, x_0)$ as our f

$p_2 : (\tilde{X}_2, \tilde{x}_2) \longrightarrow (X, x_0)$ as our p

(note that $\tilde{X}_1$ is locally path connected as p_1 restricted is a homeomorphism)

~~Fate~~

Then, we may lift ~~a~~ p_1 to a map

$$\tilde{p}_1 : (\tilde{X}_1, \tilde{x}_1) \longrightarrow (\tilde{X}_2, \tilde{x}_2)$$

with $p_2 \tilde{p}_1 = p_1$

Similarly we get $\tilde{p}_2 : (\tilde{X}_2, \tilde{x}_2) \longmapsto (\tilde{X}_1, \tilde{x}_1)$

with $p_1 \tilde{p}_2 = p_2$

By the unique lifting property

Proposition: Unique Lifting Criterion

Given a covering space $p : \tilde{X} \mapsto X$ and a map $f : Y \mapsto X$, if two lifts $\tilde{f}_1, \tilde{f}_2 : Y \mapsto \tilde{X}$ of f agree at one point of Y and Y is connected, then $\tilde{f}_1$ and $\tilde{f}_2$ agree on all of Y.

$\tilde{p}_1 \tilde{p}_2 = \mathbb{1}_{\tilde{X}_2}$ and $\tilde{p}_2 \tilde{p}_1 = \mathbb{1}_{\tilde{X}_1}$ since these composed lifts fix the basepoints.

$\therefore$ $\tilde{p}_1$ and $\tilde{p}_2$ are our inverse isomorphisms.

$\{\text{coverings of } X\} / \substack{\text{basepoint preserving} \\ \text{isomorphism}} \longleftrightarrow$ subgroups of $\pi_1(X)$

$\{\text{coverings of } X\} / \text{isomorphism} \longleftrightarrow$ conjugacy classes of subgroups of $\pi_1(X)$

$\therefore$ The universal cover is unique up to isomorphism.

Proposition: <u>Covering Space Classification Theorem</u>

Let X be path connected + locally path connected + semilocally simply connected.
Then, $\exists$ a bijection between the set of basepoint-preserving isomorphism classes of path-connected covering spaces

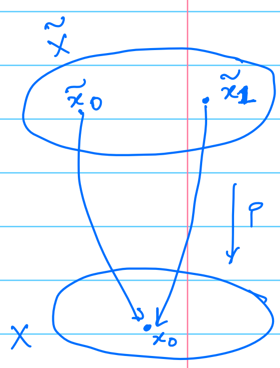
$$p : (\tilde{X}, \tilde{x}_0) \longmapsto (X, x_0)$$

and the set of subgroups of $\pi_1(X, x_0)$ obtained by associating the subgroup $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to the covering space $(\tilde{X}, \tilde{x}_0)$.

If the basepoints are ignored, this correspondence gives a bijection betn isomorphism classes of path-connected covering spaces $p : \tilde{X} \longmapsto X$ and conjugacy classes of subgroups of $\pi_1(X, x_0)$

<u>Proof</u> :

We only need to prove the last statement.
Consider the covering space $p : (\tilde{X}, \tilde{x}_0) \longmapsto (X, x_0)$. We show that changing the basepoint $\tilde{x}_0$ <u>within $p^{-1}(x_0)$</u> corresponds to changing $p_*(\pi_1(\tilde{X}, \tilde{x}_0))$ to a conjugate subgroup of $\pi_1(X, x_0)$.



$\Rightarrow$ Let $\tilde{x}_1$ be a different basepoint in $p^{-1}(x_0)$
Let $\tilde{\gamma}$ be a path from $\tilde{x}_0$ to $\tilde{x}_1$
Then, $\tilde{\gamma}$ projects to a loop γ in X representing $g \in \pi_1(X, x_0)$.

Let $H_i := p_*(\pi_1(\tilde{X}, x_i))$ for $i = 0, 1$

$\longrightarrow$ $g^{-1} H_0 g \subset H_1$ because for $\tilde{f}$ a loop at $\tilde{x}_0$, $\bar{\tilde{\gamma}} \cdot \tilde{f} \cdot \tilde{\gamma}$ is a loop at $\tilde{x}_1$

$\longrightarrow$ Similarly $g H_1 g^{-1} \subset H_0$

Now, $g^{-1}(g H_1 g^{-1}) g = H_1 \subset g^{-1} H_0 g$ (i.e. $g H_1 g^{-1} \subset H_0$)

$\therefore$ $g^{-1} H_0 g = H_1$ $\Rightarrow$ Changing the basepoint from $\tilde{x}_0$ to $\tilde{x}_1$ changes H_0 to $H_1 = g^{-1} H_0 g$, a conjugate subgroup.

$\Leftarrow$ Conversely, suppose we change H_0 to a conjugate subgroup $H_1 = g^{-1} H_0 g$. Then, choose a loop γ representing g in $\pi_1(X, x_0)$ which lifts to a path $\tilde{\gamma}$ from $\tilde{x}_0$ to $\tilde{x}_1 = \tilde{\gamma}(1)$

By the same argument above,

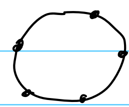
$H_1 = g^{-1} H_0 g$

Examples

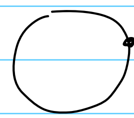
(1) $X = S^1$, $\pi_1(S) = \mathbb{Z}$

Connected covers of X $\longleftrightarrow$ Subgroups of $\mathbb{Z}$

universal cover $\mathbb{R}$ $\longleftrightarrow$ 0



$\longleftrightarrow$ $n\mathbb{Z}$ for $n \geqslant 2$



$\xleftrightarrow[\text{identity}]{}$ $\mathbb{Z}$

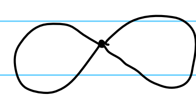
(2) $X = \mathbb{RP}^n$, $\pi_1(X) = \mathbb{Z}/2\mathbb{Z}$

subgroups of $\pi_1(X)$:

$S^n \longleftrightarrow 1$

$\mathbb{RP}^n \longleftrightarrow \mathbb{Z}_2$

(3) $X = S_1 \vee S_1$

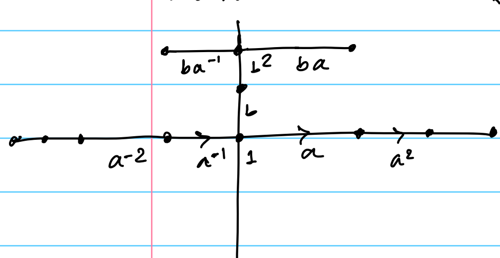


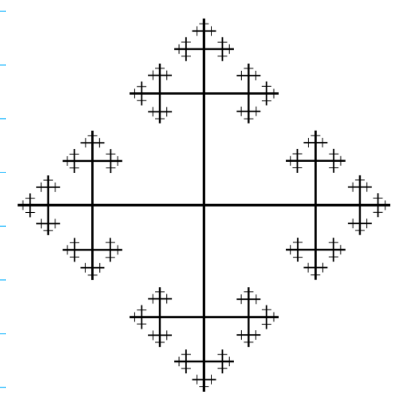
$\pi_1(S_1 \vee S_1) = \mathbb{Z} * \mathbb{Z} = F_2$

notice: it is non-abelian so lots of subgroups

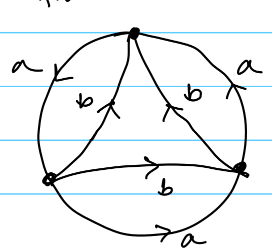
Examples of Subgroups

universal cover $\longleftrightarrow$ 1





fractal!



$\longleftrightarrow$ $H = \langle a^3, ab, ba, b^3 \rangle$

Deck Transformations

So far,

Classification of path connected covering spaces of a "good" top space $\longrightarrow$ p.c, l.p.c, sl.s.c

$$G \subset \pi_1(X) \xleftrightarrow{1:1} \tilde{X} \xrightarrow{p} X$$

$$Im(p_*) = G$$

Def Let G be a group, X - top space.

An action of G on X is a map

$$G \times X \longmapsto X$$

by $(g, x) \longrightarrow gx$

s.t $\quad e(x) = x$

$$g_1(g_2 x) = (g_1 g_2)x \quad \forall x, g_1, g_2$$

We write $\quad G \circlearrowright X$

Alternatively, we see this as a homomorphism $G \longrightarrow \underbrace{Homeo(X)}_{\text{a group}}$

Def

For $p: \tilde{X} \longmapsto X$ covering space, a deck transformation is a self-isomorphism of $\tilde{X} \xrightarrow{p} X$

(i.e isomorphism $\tilde{X} \longmapsto \tilde{X}$)

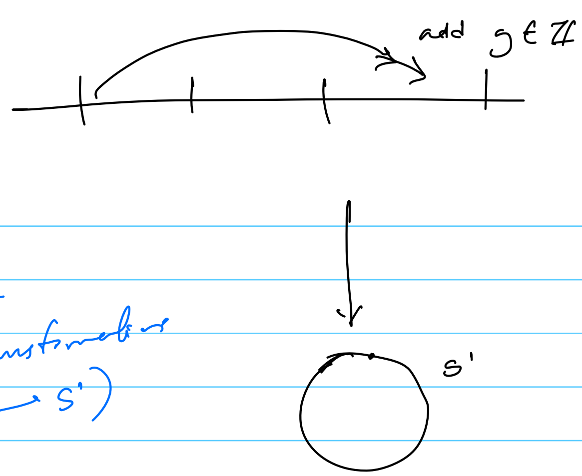
$G(\tilde{X}) \longrightarrow$ group of deck transformations (under composition)

Note: $G(\tilde{X}) \circlearrowright \tilde{X}$

By unique lifting property, a deck transformation f is uniquely determined by $f(\tilde{x}_0) \in p^{-1}(x_0)$

$$\begin{tikzcd} \tilde{x}_0 \in \tilde{X} \arrow[rr, "f"] \arrow[dr, "p"'] & & \tilde{X} \arrow[dl, "p"] \\ & X & \end{tikzcd}$$

Eg: $\mathbb{Z} \circlearrowright \mathbb{R}$ by $(g.x) = g + x \longrightarrow$ This is a deck transformation of the covering space $\mathbb{R} \longmapsto S^1$

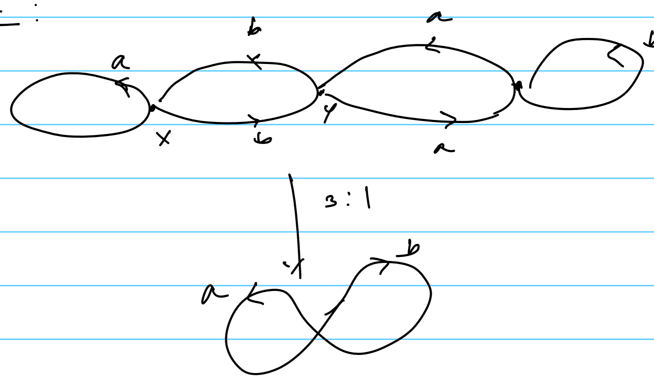


This is the entire group of deck transformations $G(\mathbb{R} \to S^1)$

Def:

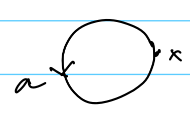
A covering space is <u>normal</u> if $\forall x_0 \in X$ and $\forall$ lifts $\tilde{x}_0, \tilde{x}_1$ of x_0, $\exists$ a desk transformation taking $\tilde{x}_0$ to $\tilde{x}_1$.

<u>Non-example</u>:



$$G = \langle a^2, b^2 ab a^{-1}, bab^{-1} \rangle \subset \langle a, b \rangle$$

There is no deck transf. taking $x \longrightarrow y$ because x has an $\underbrace{\text{'a' loop}}$ but 'y' does not.



Similarly for other points too.

Theorem: Let $H = \pi_1(\tilde{X})$, $G = \pi_1(X)$ and X is "good".

(a) $p: \tilde{X} \longmapsto X$ is normal $\iff$ $H \subset G$ is normal

i.e $gHg^{-1} = H, \; \forall g \in G$

(b) $G(\tilde{X}) = \underbrace{N(H)}/H$

$N(H)$ is the normalizer of H

i.e $N(H) = \{ g \mid gHg^{-1} = H \}$

notice $H \subset N(H)$

In particular, if $\tilde{X}$ is normal, then

$$G(\tilde{X}) = G/H.$$

Proof:

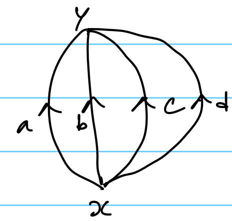
Homology Theory

Introduction

The homology group $H_n(X)$ for a CW complex X depends only on the $(n+1)$-skeleton.

Cycles

Previously, we had loops at a fixed basepoint x:



} graph X_1

So a loop at x is, for eg, ab^{-1}. This is non-abelian.
But if we abelianize this i.e $\underbrace{ab^{-1}}_{\text{loop at } x} = \underbrace{b^{-1}a}_{\text{loop at } y}$, we are rechoosing the basepoint.

ab^{-1} and $b^{-1}a$ are the same cycle since we ignore the basepoint.

$\therefore$ Abelianizing loops $\rightsquigarrow$ no longer a fixed basepoint for loops.
Rechoosing the basepoint leads to permuting the letters cyclically.
Loops $\rightsquigarrow$ <u>cycles</u>.

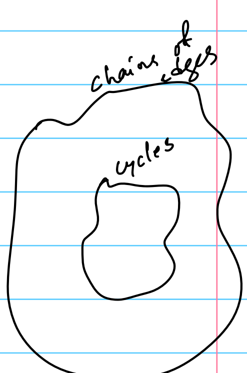
<u>Switching to Additive notation</u>:

Cycles become <u>linear combinations of edges</u> with integer coefficients like $a-b+c-b$. (instead of $ab^{-1}cd^{-1}$)

Linear combinations are called <u>chains of edges</u>.

↳ Some chains can be decomposed into cycles in many different ways

~~a+b+c+d~~ $\longrightarrow (a-c)+(b-d)$
$= (a-d)+(b-c)$



$\therefore$ We define cycle to be any chain of edges s.t $\exists$ at least one decomposition into cycles in the previous more geometric sense.

When is a chain a cycle?

A geometric cycle must ~~each~~ enter each vertex the same no. of times as it leaves the vertex.

Eg: consider the chain $ka + \ell b + mc + nd$

enters y $k+\ell+m+n$ times

leaves x $k+\ell+m+n$ times.

$\therefore$ for this to be a cycle $k+\ell+m+n$ has to be 0.

Construction of C_1, C_0 and $\partial_1 : C_1 \longmapsto C_0$:

Let C_1 be the free abelian group with the basis edges $\{a, b, c, d\}$

Let C_0 be the free abelian group with the basis vertices $\{x, y\}$.

Elements of C_1 = chains of edges or 1-dimensional chains

$(C_1 = \{a - b + c - d, a - d, b - c, \dots\})$

Elements of C_0 = linear combination of vertices or 0-dimensional chains

$(C_0 = \{x - y, y - x, x, y, \dots\})$

Define the homomorphism

$\partial_1 : C_1 \longrightarrow C_0$ s.t $\partial_1(\text{edge})$ = vertex at the head of edge − vertex at the tail of edge

s.t $\partial_1(a) = \partial_1(b) = \partial_1(c) = \partial_1(d) = y - x$

(vertex at the head of the edge) − (vertex at the tail of the edge)

$\therefore \partial_1(ka + \ell b + mc + nd) = (k+\ell+m+n)y - (k+\ell+m+n)x$

→ $X_1 \longrightarrow$ graph of these vertices and edges.

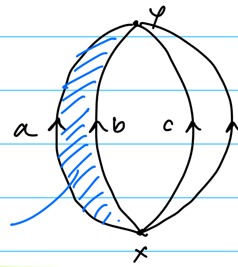
→ $\ker(\partial_1)$ = cycles

basis for $\ker(\partial_1)$: $\{a - b, b - c, c - d\}$

Construction of C_2, $\partial_2 : C_2 \longmapsto C_1$:

Now, we attach a 2-cell to our graph along the cycle $a-b$.

so, we are attaching a 2-cell along the $\ker(C_1)$.



This is our 2-dimensional cell complex X_2.

Let $C_2 = \{A, A^2, A^3, \ldots\}$ infinite cyclic group generated by A

Let $\partial_2(A) = a-b$ i.e ∂_2 (2-cell) : boundary of 2-cell.

$\therefore$ Define a pair of homomorphisms

$$C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

Construction of $H_1(X_2)$:

$\therefore$ The cycle $a-b$ is now homotopically trivial as we can slide it over A and does not enclose a hole in X_2.

$\therefore$ We quotient the group of cycles by factoring out the subgroup generated by $(a-b)$.

$\Rightarrow$ The cycles & $(a-c)$ and $(b-c)$ become equivalent since they are homotopic in X_2.

$$H_1(X_2) := \underbrace{\ker \partial_1}_{\text{one-dimensional cycles}} / \underbrace{\operatorname{Im} \partial_2}_{\text{the cycles that are boundaries of the 2-cells in } C_2 \text{ (i.e multiples of } a-b)}$$

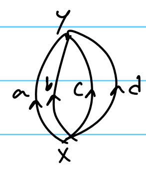
cycles that enclose holes

eg: $(a-b)$ as $\partial_1(a-b) = 0$ and $(c-d), \ldots$

i.e cycles that start & end at the same point

Examples

(i)

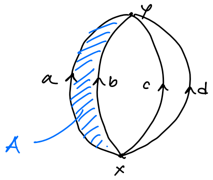


Here, $C_2 = \{0\}$.

$H_1(X_1) = \ker \partial_1 / \operatorname{Im} \partial_2 = \ker \partial_1$

$\longrightarrow$ the free abelian group on 3 generators $\{a-b, b-c, c-d\}$

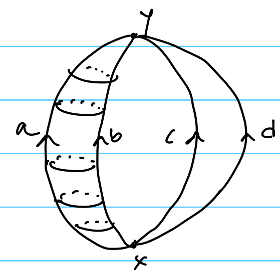
(2)



Here, $H_1(X_2)$ = free abelian group on 2 generators $\{b-c, c-d\}$

Another H_1 construction :

Now, we enlarge X_2 to a space X_3 by attaching a second 2-cell B along the same cycle $a-b$.



Now, C_2 = 2-dimensional chain group consisting of linear combinations of A and B.

$\partial_2 : C_2 \longrightarrow C_1$ sends both A and B to $(a-b)$

Then, $H_1(X_3) = \ker(\partial_1) / \underbrace{\mathrm{Im}(\partial_2)}_{\{a-b,\,(a-b)^2,\,\ldots\}} = H_1(X_2)$

still the same $\{a-b, c-d, b-d, \ldots\}$

but now ∂_2 has a non-trivial kernel which is the infinite cyclic group generated by $A-B$.

$A-B$ is the 2-dimensional cycle generating $H_2(X_3)$

$H_2(X_3) = \ker(\partial_2)/\mathrm{Im}(\partial_3) = \ker(\partial_2) \approx \mathbb{Z}$

↳ The cycle $A-B$ is the sphere formed by cells A and B with the common boundary circle.

$A-B$ detects a hole in X_3 → the interior of the sphere

$H_1(X_3)$ ⟶ detects holes enclosed by a circle modulo $\langle a-b \rangle$

i.e $\{b-c, c-d\}$

$\therefore H_1(X_3) \approx \mathbb{Z} \times \mathbb{Z}$

Now, we form X_4 from X_3 by attaching a 3-cell C along the 2-sphere formed by A and B

$\therefore$ We get a chain group C_3 and a homomorphism $\partial_3 : C_3 \longrightarrow C_2$ by sending C to $A - B$

$$C_3 \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0$$

$H_2(X_4) = \underbrace{\ker \partial_2}_{\text{generated by } (A-B)} / \underbrace{\operatorname{Im} \partial_3}_{\text{generated by } A-B}$ is trivial

as $\partial_2(A-B)$
$= \partial_2(A) - \partial_2(B)$
$= (a-b) - (a-b) = 0$

$H_3(X_4) = \ker \partial_3 = 0$

$H_1(X_4) = H_1(X_3) \approx \mathbb{Z} \times \mathbb{Z}$

Generally,
for a cell complex X, we have chain groups $C_n(X)$ which are free abelian groups with basis the n-cells of X and there are boundary homomorphisms

$$\partial_n : C_n(X) \longrightarrow C_{n-1}(X)$$

$\partial_1 \longrightarrow$ boundary of an oriented edge is (vertex at its head) $-$ (vertex at its tail)

$\partial_2 \longrightarrow$ boundary of a 2-cell attached along a cycle is the cycle of edges.

What about ∂_n for $n > 2$?

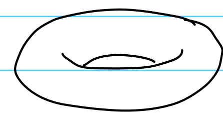
Simplical and Singular Homology

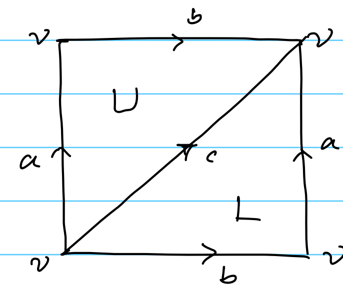
Δ-complexes

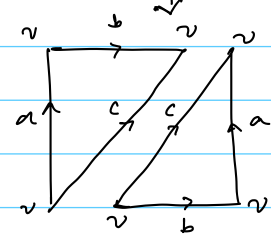
Motivation :

We can form the torus, $\mathbb{RP}^2$, klein bottle from a square:

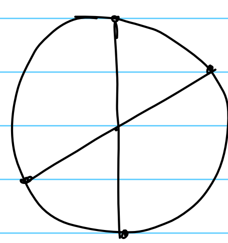
T :

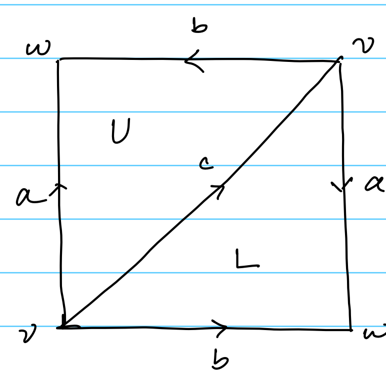




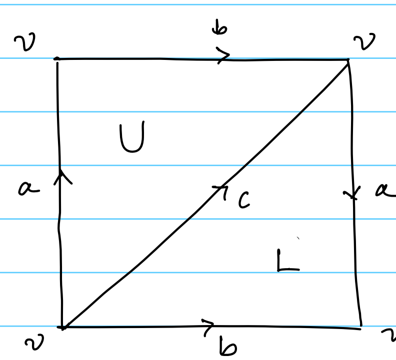


$\mathbb{RP}^2$:





K :



A polygon with any no. of sides can be cut along diagonals into triangles, so <u>all closed surfaces can be constructed from triangles by identifying edges</u>.

Def : n- simplex

Smallest convex set in $\mathbb{R}^m$ containing $(n+1)$ points $v_0, \ldots, v_n \in \mathbb{R}^m$ that do not lie in a hyperplane of dimension less than m

↳ a set of solutions of a system of linear equations.

Hyperplane in $\mathbb{R}^m$ has dimension $m-1$.

Eg: for $n=2$ (a triangle), the points must not lie on a single line (a 1-dimensional hyperplane)

$\Updownarrow$

The difference vectors $v_1 - v_0, \ldots, v_n - v_0$ are linearly independent.

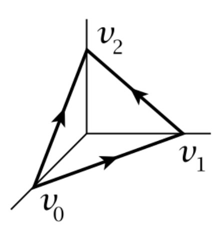
Vertices $\longrightarrow$ v_i

The simplex is denoted by $[v_0, \ldots, v_n]$.

Example :

(i) The standard n-simplex

$$\Delta^n := \left\{ (t_0, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{i=0}^{n} t_i = 1 \text{ and } t_i \geq 0 \;\forall i \right\}$$



Ordering the vertices in an n-simplex

We have the ordering $[v_0, \ldots, v_n]$

→ Determines an orientation of the vertices $[v_i, v_j]$ according to increasing subscripts.

→ This also gives us a homeomorphism from the standard n-simplex Δ^n onto any other n-simplex $[v_0, \dots, v_n]$ preserving the order of the vertices

$$\varphi : \Delta^n \longmapsto [v_0, \dots, v_n]$$

by $$\varphi((t_0, \dots, t_n)) = \sum_{i=0}^{n} t_i v_i$$

→ we call the coefficients t_i the <u>barycentric coordinates</u> of the point $\sum_i t_i v_i$ in $[v_0, \dots, v_n]$.

<u>Face</u>

If we delete one of the $n+1$ vertices in $[v_0, \dots, v_n]$, then the remaining $n-1$ vertices span an $(n-1)$ simplex called a <u>face</u> of $[v_0, \dots, v_n]$.

→ The vertices of a face or of any subsimplex spanned by a subset of the vertices will always be ordered according to their order in the larger simplex.

Boundary

The union of all the faces of Δ^n is called the boundary of Δ^n, denoted by $\partial\Delta^n$

Open Simplex

The open simplex $\mathring{\Delta}^n = \Delta^n - \partial\Delta^n$ is the interior of Δ^n.

Δ - complex

A "Δ - complex" structure on a space X is a collection of maps $\sigma_\alpha : \Delta^n \longmapsto X$ with n depending on the index α s.t

(1) The restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective and each point of X is in the image of exactly one such restriction $\sigma_\alpha|_{\mathring{\Delta}^n}$

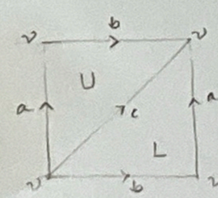
(2) The restriction of σ_α to a face of Δ^n is one of the maps $\sigma_\beta : \Delta^{n-1} \longrightarrow X$

(3) The set $A \subset X$ is open if and only if $\sigma_\alpha^{-1}(A)$ is open in Δ^n for each σ_α.

Examples

(1) Consider the following constructions

Torus, T :



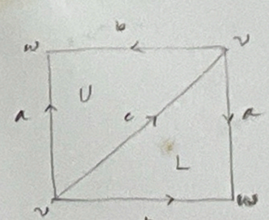
$\longrightarrow$ Six σ_α's

(1) 2 for the 2-simplices U and L

(2) 3 for the 1-simplices a, b and c

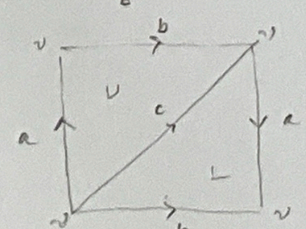
(3) 1 for the 0-simplex v (the vertex)

$\mathbb{RP}^2$:



$\longrightarrow$ 7 σ_α's.

Klein bottle, K :



$\longrightarrow$ Six σ_α's

Simplicial Homology

Let X be a Δ-complex → collection of maps $\sigma_\alpha : \Delta^n \to X$ where n & α vary.

Let $\Delta_n(X)$ be the free Abelian group with basis the open n-simplices e^n_α of X.

→ for set $S = \{s_1, s_2, \dots\}$
$\mathbb{Z}[S] = \{ \underbrace{n_1 s_1 + \dots + n_k s_k}_{} : n_i \in \mathbb{Z} \}$
only finitely many n_i are non-zero

└ Elements of $\Delta_n(X)$ are called n-chains and can be written as finite formal sums $\sum_\alpha n_\alpha e^n_\alpha$ where $n_\alpha \in \mathbb{Z}$.

$\Updownarrow$ (equivalently, define using maps describing the simplices)

Elements of $\Delta_n(X)$ can be written as $\sum_\alpha n_\alpha \sigma_\alpha$ where $\sigma_\alpha : \Delta^n \to X$ is the characteristic map of e^n_α with image the closure of e^n_α

So,
→ $\Delta_0(X)$ is generated by the vertices of X
→ $\Delta_1(X)$ is generated by the edges of X
→ $\Delta_2(X)$ is generated by triangles in X.

→ the boundary of the n-simplex $[v_0, \dots, v_n]$ consists of various $(n-1)$-dimensional simplices $[v_0, \dots, \hat{v}_i, \dots, v_n]$
↳ this vertex is deleted

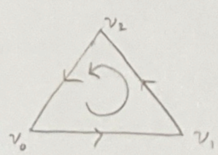
∴ The boundary ~~=~~ of $[v_0, \dots, v_n]$ is the $(n-1)$-chain formed by the ~~sum of the~~ sum of the faces $[v_0, \dots, \hat{v}_i, \dots, v_n]$.

But better to write with signs:

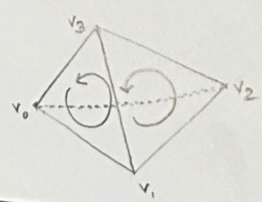
The boundary of $[v_0, \dots, v_n]$ is $\sum_{i=0}^{n} (-1)^i [v_0, \dots, \hat{v}_i, \dots, v_n]$

Examples:

$\partial [v_0, v_1] = [v_1] - [v_0]$



$\partial [v_0, v_1, v_2] = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$



$\partial [v_0, v_1, v_2, v_3] = [v_1, v_2, v_3] - [v_0, v_2, v_3] + [v_0, v_1, v_3] - [v_0, v_1, v_2]$

Boundary homomorphism

NOT Δ^n

for a general Δ-complex X, the boundary homomorphism

$$\partial_n : \Delta_n(X) \longrightarrow \Delta_{n-1}(X)$$

$$\text{s.t } \partial_n(\sigma_\alpha) = \sum_{i=0}^{n} (-1)^i \sigma_\alpha \big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

represents the n-simplex in X

Lemma:

The composition $\Delta_n(X) \xrightarrow{\partial_n} \Delta_{n-1}(X) \xrightarrow{\partial_{n-1}} \Delta_{n-2}(X)$ is zero. i.e $\partial_{n-1} \circ \partial_n : \Delta_n(X) \longrightarrow \Delta_{n-2}(X)$ is zero.

"boundary of the boundary is zero" Becomes zero because of orientations.

Proof:

$$\partial_n(\sigma) = \sum_i (-1)^i \sigma \big|_{[v_0, \dots, \hat{v}_i, \dots, v_n]}$$

$$\therefore \partial_{n-1}\partial_n(\sigma) = \sum_{j<i} (-1)^i (-1)^j \sigma \big|_{[v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n]}$$

$$+ \sum_{j>i} (-1)^i (-1)^{j-1} \sigma \big|_{[v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n]}$$

$$= 0$$

Simplicial Homology

So far, we have:

$$\cdots \; C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \longrightarrow \cdots\cdots \longrightarrow C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0$$

where $\partial_n \partial_{n+1} = 0$ for each n.

Now, $\partial_n \partial_{n+1} = 0$ is equivalent to saying $\operatorname{Im} \partial_{n+1} \subset \ker \partial_n$

n^{th} Homology Group $\longrightarrow$ $H_n = \ker \partial_n / \operatorname{Im} \partial_{n+1}$

Elements of $\ker \partial_n$ $\longrightarrow$ called cycles

Elements of $\operatorname{Im} \partial_{n+1}$ $\longrightarrow$ called boundaries

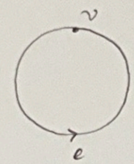
Elements of H_n (ie cosets of $\operatorname{Im} \partial_{n+1}$) $\longrightarrow$ called homology classes

Two cycles representing the same homology class are called homologous.

n^{th} Simplicial Homology Group $\longrightarrow$ $C_n = \Delta_n(X)$. Then, $H_n^\Delta(X) := \ker \partial_n / \operatorname{Im} \partial_{n+1}$

Examples of Simplicial Homology Groups

(1) Let $X = S^1$ (so 1 vertex + 1 edge)



Now, $\Delta_0(S^1)$ is generated by just the vertex v

so, $\Delta_0(S^1) \approx \mathbb{Z}$

$\Delta_1(S^1)$ is generated by edge e

so, $\Delta_1(S^1) \approx \mathbb{Z}$

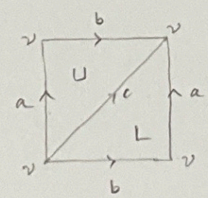
$\cancel{\partial} \; \partial_1 : \Delta_1(S^1) \longrightarrow \Delta_0(S^1)$ is 0 as

$\partial(e) = v - v = 0$

$\Delta_n(S^1)$ for $n \geqslant 2$ are zero groups since there are no simplices in these dimensions in S^1.

$$\therefore \; H_n^{\Delta}(S^1) \approx \begin{cases} \mathbb{Z} & \text{for } n = 0, 1 \\ 0 & \text{for } n \geqslant 2 \end{cases}$$

(2) Let $X = T$, the torus. T :



Again $\partial_1 = 0$ as the boundary of each edge has the same points on both ends.

$\therefore H_0^{\Delta}(T) \approx \mathbb{Z}$

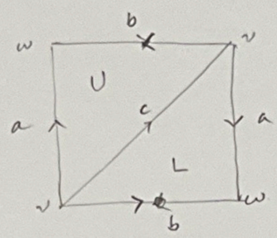
Now, $\partial_2 U = a + b - c = \partial_2 L$ and $\{a, b, a+b-c\}$ is a basis for $\Delta_1(T)$.

$\therefore H_1^{\Delta}(T) \approx \mathbb{Z} \oplus \mathbb{Z}$ with basis the homology classes $[a]$ and $[b]$.
~~(as c = a+b)~~ (as in $H_1^{\Delta}(T)$,
$\mathrm{Im}\, \partial_2 = 0$ so $a + b - c = 0$. so $c = a + b$)

Since there are no 3-simplices,
$H_2^{\Delta}(T) = \ker \partial_2 \longrightarrow$ infinite cyclic generated by $U - L$
as $\partial(pU + qL) = (p+q)(a+b-c) = 0$ iff $p = -q$

$$\therefore \; H_n^{\Delta}(T) = \boxed{\begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n = 1 \\ \mathbb{Z} & n = 0, 2 \\ 0 & n = 3, \ldots \end{cases}}$$

(3) Let $X = \mathbb{RP}^2$



$\text{Im}\, \partial_1$ = generated by $w - v$

$H_0^\Delta(X) \approx \mathbb{Z}$

Now, $\partial_2 U = -a + b + c$

$\partial_2 L = a - b + c$

So, ∂_2 is injective

$H_2^\Delta(X) = 0$

Also, $\ker \partial_1 \approx \mathbb{Z} \oplus \mathbb{Z}$ with basis $a-b$, and c

and $\text{Im}\, \partial_2$ is an index 2 subgroup of $\ker \partial_1$ as we can choose c and $a-b+c$ as a basis for $\ker \partial_1$

and $\{a-b+c,\ 2c = (a-b+c) + (-a+b+c)\}$ as a basis for $\text{Im}\, \partial_2$

$\therefore\ H_1^\Delta(X) \approx \mathbb{Z}_2$